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OF
EUCLID,

DESIGNED TO MEET THE REQUIREMENTS OF UNIVERSITY AND COLLEGIATE STUDENTS, THE NATIONAL TEACHERS, INTERMEDIATE EDUCATION STUDENTS, ETC., ETC.

EDITED BY
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Author of the famous "Task Book Series," for Schools; "Agricultural Home Lesson Series;" "Grammar Sheets" and "Geography Sheets" for Collective Teaching, etc., etc.

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PREFACE.

THE difficulties attending Examinations in Euclid are admitted to be every day increasing. We have in various Programmes certain portions of the subject specified, but the Examiners, to the great perplexity of Students, never limit themselves to the prescribed pages of the text, rather making it an apparent hobby to puzzle the Scholar upon the broad and seeming unlimited field for exercises on the subject.

It is to supply the want for a book which will help the Student to cope with this serious obstacle to promotion that the present work is published. With that view the number of Exercises here given is by far the largest which has hitherto been published in any country upon the Third, Fourth, Fifth, and Sixth Books of Euclid, so as to give the Student a complete mastery over the extensive range which Examiners are wont to travel.

Every Exercise worthy of note proposed for solution in any important work upon the subject, or which has been given at Competitive Examinations, will be found in these pages, the solution of someof which we presume to be the clearest and most scientific that has yet appeared, many being worked in two or more ways so as to give the Student a selection, and, at the same time, aid his powers of comparison and invention. It will be observed that originality is a feature of the work for which credit may be taken, as some of the most difficult.

Exercises are for the first time divested of the cumbrous machinery of construction in which they are usually clouded, and are here demonstrated in a style which, we trust, will at once recommend itself for simplicity and originality.

In the selection, those Exercises which may be considered as the essence of Euclid are all accorded a place; while the arrangement has been to place them, as far as consistent, in the order of their comparative difficulty, at the same time grouping propositions in families so as to afford greater facilities for acquiring the analysis of the subject, and to help the memory.

The separation of the Cuts from the Text, and placing them together in convenient plates, has been approved by eminent educationists, it being obvious that more scope will be thereby given for attempting the questions as *Test Papers* without the help which would necessarily follow if the Cuts were in the same page with the enunciations.

P. M. E.

EXERCISES UPON EUCLID.

THIRD BOOK.

EXERCISE 1.

If two circles cut each other, the line joining the points of intersection is bisected at right angles by the line joining their centres.

(Fig. 1, Plate I.)—Let ABR, ABL be two circles intersecting in the points A and B; from the centres M and N draw MK, NK to the point of bisection of the common chord AB.

In the circle ABR, because the line MK is drawn from the centre to the middle point of the chord AB, it therefore cuts it at right angles **(III. 3)**; for a similar reason NK bisects AB at right angles, and $\therefore$ MK, NK are in one and the same straight line **(I. 14)**.

EXERCISE 2.

AB is the diameter and C the centre of a semicircle: show that K the centre of any circle inscribed in the semicircle is equidistant from C and from the tangent to the semicircle parallel to AB.

(Fig. 2, Plate 1.)—Produce DK to G, and join CF.

Because the circles touch, the straight line joining their centres C and K passes through their point of contact E **(III. 11)**; and since $CF = CE$, and $CF = DG$, $\therefore$ $CE = DG$; but $DK = EK$, and, taking these equals from the equals CE and DG, the remainder CK = the remainder GK.

EXERCISE 3.

Through a given point within a circle, to draw a chord which shall be bisected in that point, and prove it to be the least.

(Fig. 3, Plate I.)—Let N be the centre, and ANB the diameter through the given point P; MPK a chord at right angles to AB: then MK is bisected at the point P, and is also the least chord that can be drawn through that point.

MK is bisected in the point P (III. 3), and is less than any other chord RPS (III. 15), because the perpendicular NT is less than NP (I. 19).

EXERCISE 4.

If a circle be described on the radius of any circle, any straight line drawn from the point where they meet to the outer circumference is bisected by the interior one.

(Fig. 4, Plate I.)—Let BCA be a circle described on the radius AB of the circle BDM, BD any straight line meeting the outer circumference in D and cutting the inner arc in C: then $BC = CD$.

Join CA.

BCA is a right angle (III. 31), $\therefore BC = CD$ (III. 3).

EXERCISE 5.

The straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

(Fig. 5, Plate I.)—Let $ABCD$ be a quadrilateral inscribed in the circle $ABCD$, and F, G, H, K their central points; it is evident (III. 1) that the perpendiculars from these points all pass through the centre of the circle, and must therefore intersect in that point.

EXERCISE 6.

A quadrilateral is described so that its sides touch a circle: show that the sums of the opposite sides are equal.

(Fig. 6, Plate I.)—Let $ACEG$ be the required quadrilateral, its sides touching the circle $BDFH$ in the points B, D, F, H: then $AC + GE = AG + CE$.

Because $AH = AB$ (*Schol.* III. 17), $HG = GF$, $ED = EF$, $CD = CB$, then $AH + HG + ED + DC = AB + BC + GF + FE$, that is, $AG + CE = AC + GE$.

EXERCISE 7.

C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA; join AB and produce it to meet the circle again at D, and let the tangent at D meet CB produced at E: show that BDE is an isosceles triangle.

(Fig. 7, Plate I.)—Produce AC to the circumference at F, and join DF.

Because the two triangles FDA and CBA have each a right angle and a common angle, they are equiangular (I. 32), and the third angles DFA and CBA are equal; but $CBA = DBE$ (I. 15), and $EDB = DFA$ (III. 32); $\therefore EDB = EBD$ (*Ax.* 1), and $ED = EB$ (I. 6), hence BDE is isosceles.

EXERCISE 8.

OA, OB, OC are three chords of a circle; the angle AOB is equal to the angle BOC, and OA is nearer to the centre than OB. From B a perpendicular is drawn on OA, meeting it at P, and a perpendicular on OC produced, meeting it at Q: show that AP is equal to CQ.

(Fig. 8, Plate I.)—Because the angle AOB = the angle BOC, the arc AB = arc BC (III. 26), and the chord AB = chord BC (III. 29). Again, because the angle BOP = angle BOQ (*Hyp.*), and the angles at P and Q right angles, and OB common to both the triangles BOP and BOQ, the perpendicular BP = the perpendicular BQ (I. 26). But as $AB = BC$, and $BP = BQ$, $\therefore AP = CQ$ (*Cor.* I. 47).

EXERCISE 9.

A is any point in the diameter produced of a circle whose centre is K; KM a radius perpendicular to the diameter: if AM cut the circle at R, and the tangent at R cut KA at S, show that $SA = SR$.

(Fig. 9, Plate I.)—Join KR.

$KMA + KAM = KRM + SRA$, each being equal to a right angle, but $KMA = KRM$; $\therefore SAR = SRA$, hence the sides SA, SR are equal.

EXERCISE 10.

Two circles cut one another, and two diameters are drawn which meet in one of the points of intersection : show that the straight line which joins the other extremities of the diameters passes through the other point of intersection.

(Fig. 10, Plate I.)—Let *BKA*, *CKA* be two circles, intersecting in the points *A*, *K*, of which *AB*, *AC* are the respective diameters : then the straight line joining *BC* passes through *K*.

Join *BK*, *AK* and *CK*.

Because *BKA* is a semicircle (*Hyp.*), ∴ *BKA* is a right angle (**III. 31**), and, for a similar reason, *CKA* is a right angle ; ∴ *BK* and *CK* are in the same straight line (**I. 14**).

EXERCISE 11.

To draw a line from a given point A such that the perpendicular on it from a given point B shall be equal to a given line BM.

(Fig. 11, Plate I.)—With *B* as centre, and the given line *BM* as radius, describe the circle *MNR* ; from *A* draw *AM*, a tangent to *MNR* (**III. 17**) : *AM* is the line required.

Join *BM*.

Because *AM* is a tangent, *AMB* is a right angle (**III. 19**), and *BM* equals the given line.

EXERCISE 12.

If AD, CE be drawn perpendicular to the sides BC, AB of a triangle ABC, and DE be joined : show that the angles ADE, ACE are equal to each other.

(Fig. 12, Plate I.)—Because *AEC*, *ADC* are right angles they are ∴ the angles in a semicircle *AEDC*, of which *AC* is the diameter ; hence the angles *ADE*, *ACE* are equal (**III. 21**).

EXERCISE 13.

If two equal circles intersect, and if through one of the points of intersection a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

(Fig. 13, Plate I.)—Let *MAC* be a straight line drawn through *A*, one of the points of intersection; join the points *M*, *C* with the other point of intersection *B*: *BM* = *BC*.

Join *AB*.

Because the circles are equal the intercepted arcs are equal (III. 28); therefore the angles at *M* and *C* are equal (III. 26); and hence *BM* and *BC* are equal (I. 5).

EXERCISE 14.

On the side AB of any triangle ABC as diameter a circle is described; EF is a diameter parallel to BC: show that the straight lines EB and FB bisect the interior and exterior angles at B.

(Fig. 14, Plate I.)—Because *EF* is parallel to *CB* and *EB* cuts them, ∴ *FEB* = *EBC* (I. 29); but since the radii *LE*, *LB* are equal, the angles *FEB*, *LBE* are equal (I. 5), whence the angles *EBC*, *LBE* are equal. It may be similarly shown that the exterior angle is also bisected.

EXERCISE 15.

AD and BE are perpendiculars from the angles A and B of a triangle on the opposite sides; BF is perpendicular to ED or ED produced: show that the angle FBD is equal to the angle EBA.

(Fig. 15, Plate I.)—Because the angles *AEB* and *ADB* are right angles, *E* and *D* are in the circumference of the circle described on *AB* as diameter (III. 31).

Again, the angle *DAB* = angle *BEF* (III. 21); and the angles *ADB* and *EFB* are right angles; therefore the third angles *ABD* and *FBE* of the triangles *ADB* and *EFB* are equal (I. 32). From each of these equals take the common angle *EBD*, and the remaining angles *EBA* and *FBD* are equal (*Ax.* 3).

EXERCISE 16.

Two circles intersect at A and B. At A the tangents AC and AD are drawn to each circle and terminated by the circumference of the other. If CB and BD be joined, show that AB. produced if necessary, bisects the angle CBD.

(**Fig. 16, Plate II.**)—Produce DA and CA to F and G. The angle CAF = angle DAG (I. 15); the angle CAF = angle ABC (III. 32), and, for the same reason, the angle DAG = angle ABD; ∴ the angle ABC = angle ABD (*Ax.* 1), hence (I. 13 and *Ax.* 3) the angle CBE = angle DBE.

EXERCISE 17.

AB is any chord, and AD is a tangent to a circle at A. DPQ is any straight line parallel to AB, meeting the circumference at P and Q: show that the triangle PAD is equiangular to the triangle QAB.

(**Fig. 17, Plate II.**)—Draw AP, BQ, and AQ. The angle DAP = angle AQP (III. 32), and the angle BAQ = angle AQP (I. 29); ∴ the angle DAP = angle BAQ (*Ax.* 1). Again, the angles ABQ and APQ are equal to two right angles (III. 22), and the angles APD and APQ are also equal to two right angles (I. 13); ∴ the angle ABQ = angle APD (*Ax.* 3); therefore the third angles ADP and AQB are equal (I. 32), and the triangle PAD is equiangular to the triangle QAB.

EXERCISE 18.

Two circles, ABDH and ABG, intersect each other at the points A and B; from B a straight line BD is drawn in the one to touch the other; and from A any chord whatever is drawn cutting the circles at G and H: show that BG is parallel to DH.

(**Fig. 18, Plate II.**)—The angle ABD = angle AGB (III. 32), and the angle AHD = angle ABD (III. 21); ∴ the angle AGB = angle AHD (*Ax.* 1), and hence (I. 27) the line BG is parallel to DH.

EXERCISE 19.

AB is any chord of a circle, P any point in the circumference; PM is a perpendicular on AB and is produced to meet the circle at Q; and AN is drawn perpendicular to the tangent at P: the triangles NAM, PAQ are equiangular.

(**Fig. 19, Plate II.**)—ANP, AMP are right angles, and ∴ $ANPM$ is inscribable in a circle, whence the angles AMN, APN are equal, but $APN = AQP$ (III. 32); ∴ $AMN = AQP$; and ANM = APM; hence the angles NAM, PAQ are equal (I. 32), and the triangles NAM, PAQ are equiangular.

EXERCISE 20.

AOB, COD are diameters of a circle at right angles to each other; E is a point in the arc AC, and EFG is a chord meeting COD at F, and drawn in such a direction that EF is equal to the radius: show that the arc BG is three times the arc AE.

(Fig. 20, Plate II.)—Join EO, EB.

Because FEO is isosceles, $\therefore$ half the angle FEO together with FOE equals the angle COA, or a right angle; from each take the common part FOE, and the whole angle at E is double EOA; but EOA is double of OEB; $\therefore$ GEB equals thrice OEB, or OBE, whence the arc GB is three times the arc AE.

EXERCISE 21.

Two circles intersect at A and B, and through P any point in the circumference of one of them the chords PA, PB are drawn to cut the other circle at D and C: CD is parallel to the tangent MPN.

(Fig. 21, Plate II.)—Join AB.

Angle $NPC = PAB$ (III. 32) $= DCB$ (I. 13 and III. 22); $\therefore$ CD is parallel to MPN (I. 27).

EXERCISE 22.

It is required to find a point in the straight line which touches a circle at the end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

(Fig. 22, Plate II.)—Let MN be the given diameter, NR the perpendicular at one extremity, and NR the line whose square is given.

With N as centre, and NR as radius, describe an arc cutting the circumference in K; join MK, and produce it to cut NR in L: L is the point required.

Join NK.

$ML \cdot LK = LN^2$ (III. 36) $= KL^2 + KN^2$, but $ML \cdot LK = MK \cdot KL + KL^2$; $\therefore$ $KL^2 + KN^2 = MK \cdot KL + KL^2$; take away the common KL^2, and $KN^2 = MK \cdot KL$.

EXERCISE 23.

From each of the angular points of a triangle a perpendicular is let fall on the opposite side: show that the rectangles contained by the segments into which each perpendicular is divided by the point of intersection of the three are equal to each other.

(**Fig. 23, Plate II.**)—Let the perpendiculars BD, AE, CF be drawn to the sides AC, CB, and AB respectively intersecting in K.

$AK \cdot KE = CK \cdot KF = BK \cdot KD$.

Since the angles at F and E are right angles, AC is the diameter of a circle passing through F and E, and $\therefore$ $AK \cdot KE = CK \cdot KF$ (III. 35); for a similar reason $BK \cdot KD = CK \cdot KF$; $\therefore$ $AK \cdot KE = CK \cdot KF = BK \cdot KD$.

EXERCISE 24.

Three circles touch one another externally at the points A, B, C; from A, the straight lines AB, AC are produced to cut the circle BCH at D and E: show that DE is a diameter of BCH, and is parallel to the straight line joining the centres of the other circles.

(**Fig. 24, Plate II.**)—Join K and O, K and P, O and P, the three centres, by the lines KO, KP, OP, which must pass through the points of contact A, B, C (III. 12), and join D and P, E and P: then DPE is one straight line, and is therefore a diameter of the circle $DBCEH$.

Because $KA = KB$, the angle KAB = angle KBA; and because $PB = PD$, the angle PBD = the angle PDB (I. 5); but the angle PBD = angle KBA (I. 15); $\therefore$ the angle KAB = angle PDB (*Ax.* 1), and KAO is parallel to PD (I. 27). In like manner it may be shown that the angle OAC = angle PEC, hence OAK is parallel to PE, and PD is parallel to KAO; $\therefore$ DPE is one straight line, and a diameter. It could also be thus shown to be one straight line:—KAO is parallel to PD, $\therefore$ angle BKA = angle DPB (I. 29); and angle at O = angle EPC; to these equals add angle BPC; and $\therefore$ (I. 32, 14) DPE is one straight line.

EXERCISE 25.

The greatest rectangle which can be inscribed in a circle is a square.

(**Fig. 25, Plate II.**)—Let $ADBE$ be a square inscribed in the circle $ADFBE$, and let $AFBG$ be any other rectangle inscribed in the circle : then $ADBE$ is greater than $AFBG$.

From F draw FHK perpendicular to AB, and join DE.

Because the diameter is the greatest chord in a circle (III. 15), DE is greater than FK, and DC than FH ; ∴ the triangle ADB is greater than the triangle AFB, and the double of ADB is greater than the double of AFB—that is, the square $ADBE$ is greater than the rectangle $AFBG$, and ∴ it is greater than any other rectangle.

EXERCISE 26.

Two equal circles touch each other externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other : show that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

(**Fig. 26, Plate II.**)—Let APD and BPE be two equal circles touching each other externally in the point P ; and let PA and PB be two chords, one in each circle, at right angles to each other, meeting the circles in A and B : then the straight line joining A and B is equal and parallel to CK, the straight line joining the centres of the circles.

Join A and C, B and K.

Because the angle APB is (*Hyp.*) a right angle, the angles APC and BPK are equal to one right angle, and their equals CAP and KBP are also equal to one right angle ; ∴ the four angles CAP, CPA, KBP, and KPB are equal to two right angles, hence the angles ACP and BKP are together equal to two right angles (I. 32) ; ∴ AC is parallel to BK (I. 29), and hence (I. 33) AB is equal and parallel to CK.

EXERCISE 27.

B is a point in the circumference of a circle, whose centre is C; PA, a tangent at any point P meets CB produced at A, and PD is drawn perpendicular to CB : show that the straight line PB bisects the angle APD.

(**Fig. 27, Plate II.**)—Because the two triangles APC and APD have each a right angle and a common angle at A, they are equiangular (I. 32), and the angle ACP = angle APD

(*Ax.* 3). Again, the angle APB is equal to half of the angle ACP (III. 32 and 20); ∴ the angle DPB is also equal to half the angle ACP (*Ax.* 3), hence the angle APB = angle DPB (*Ax.* 1); ∴ the angle APD is bisected by the line BP.

EXERCISE 28.

If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars drawn to them from the middle point of the subtended arc are equal to each other.

(Fig. 28, Plate II.)—Let AB be the chord and BC the tangent drawn from any point B; let the arc ADB be bisected in D and the perpendiculars DE, DC be drawn to the chord and tangent respectively: then $DE = DC$.

Join AD, BD.

Now $AD = BD$ (III. 29); and angle DAB = angle DBC (III. 32); ∴ triangles DEA, DCB are equal (I. 26), and hence $DE = DC$.

EXERCISE 29.

Two circles intersect in the points A and B; through A and B any two straight lines CAD, EBF, are drawn cutting the circles in the points C, D, E, F: prove that CE is parallel to DF.

(Fig. 29, Plate II.)—Join AB.

Because $ECAB$ is a quadrilateral inscribed in a circle, the angles ECA, FBA are equal; for a similar reason $EBA = FDA$, but $EBA + FBA$ = two right angles; ∴ $ECA + FDA$ = two right angles, whence CE is parallel to DF (I. 28).

EXERCISE 30.

In a circle, the sum of the squares of two lines drawn from the extremities of a chord, to any point in a diameter parallel to it, is equal to the sum of the squares of the segments of the diameter.

(Fig. 30, Plate II.)—Let SR be the chord; LD the parallel diameter; SK, RK the lines drawn from the extremities of the chord to any point K in LD: then $SK^2 + RK^2 = LK^2 + DK^2$.

From the radius MD cut off $MN = MK$, and join MS, MR, RN.

Triangles KSM, NRM are equal (I. 4), ∴ $SK = RN$. Now, $SK^2 + RK^2 = NR^2 + RK^2 = 2KM^2 + 2MR^2 = 2KM^2 + 2MD^2 = LK^2 + DK^2$ (II. 9).

PLATE I.

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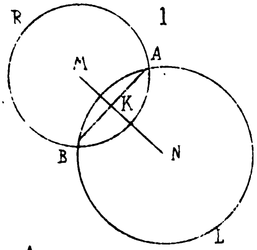

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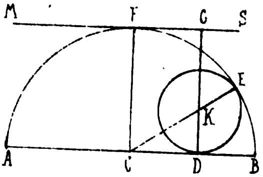

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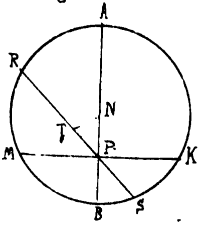

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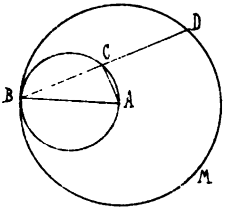

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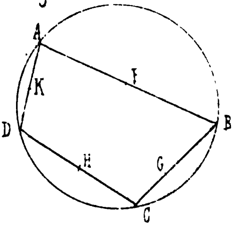

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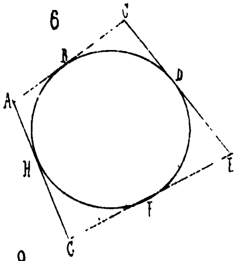

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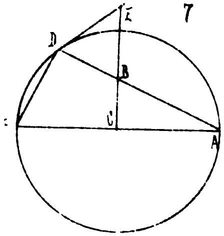

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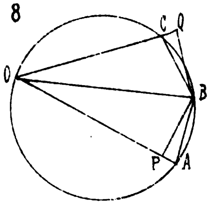

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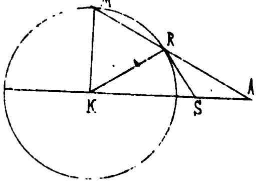

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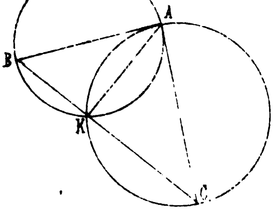

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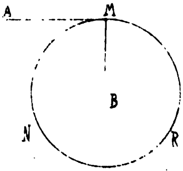

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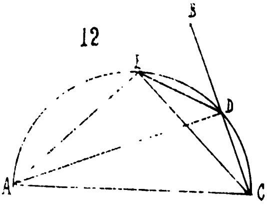

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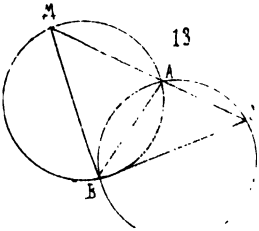

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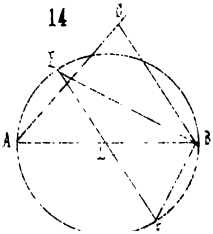

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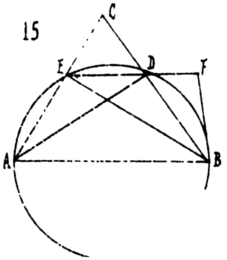
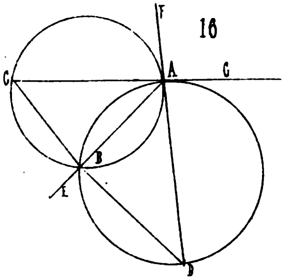


PLATE II.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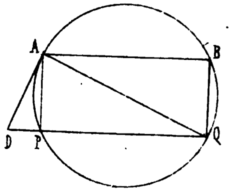
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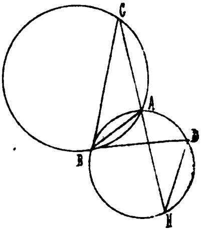
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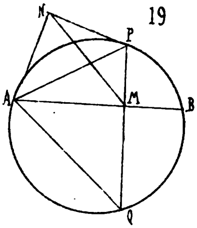
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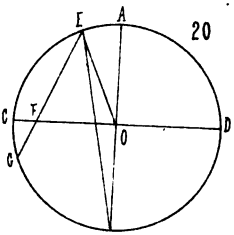
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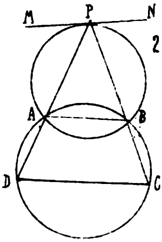
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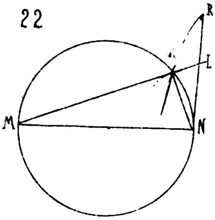
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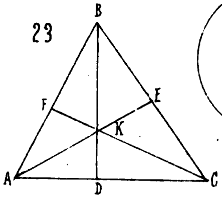
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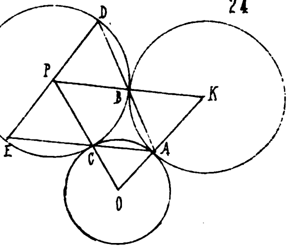
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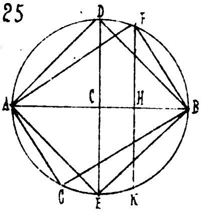
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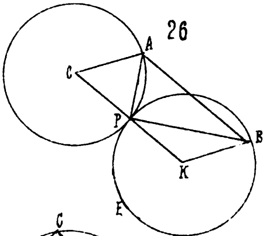
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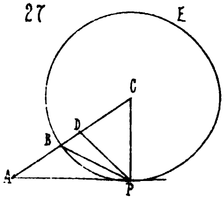
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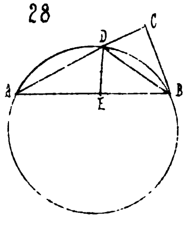
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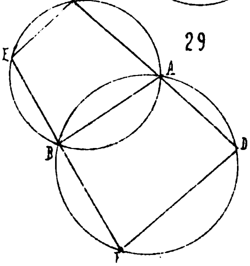
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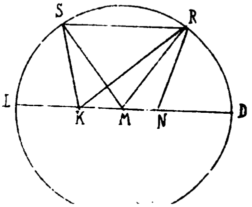
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EXERCISE 31.

If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected; the line joining the points of bisection will be trisected by the sides.

(Fig. 31, Plate III.)—Let ABC be an equilateral triangle inscribed in a circle; DE the line joining the middle points of the adjacent arcs AB, BC: DE is trisected in the points L, M, where it intersects the sides of the triangle.

Join DB, EB.

Because DE is parallel to AC, LBM is an equilateral triangle; and because the angles ABD, BDE, CBE, DEB are equal (III. 27), $\therefore DL = BL = LM$, and $EM = BM = LM$; and hence DE is trisected in the points L, M.

EXERCISE 32.

The triangle formed by the chord of a circle, the tangent at its extremity, and any line perpendicular to the diameter drawn through its other extremity, will be isosceles.

(Fig. 32, Plate III.)—Let AN be the chord, MN the tangent, and AB, RM the diameter and perpendicular respectively, and let the perpendicular meet the tangent at M cutting the chord at K: KMN is isosceles.

Join BN.

Because the angles KRB, BNA are right angles, the triangles AKR, ABN are equiangular (I. 32), and $MNK = NBA$ (III. 32) $= AKR = MKN$, $\therefore KMN$ is isosceles (I. 6).

EXERCISE 33.

Two circles touch each other externally, the diameter of one being double of the diameter of the other; through the point of contact any line is drawn to meet the circumference of both: show that the part of the line which lies in the larger circle is double of that in the smaller.

(Fig. 33, Plate III.)—Let SNM be the smaller circle touching the larger at S, and RSN the line through the point of contact: KS is double of SN.

Produce the diameter MS of the smaller circle to the centre W of the larger; join MN and draw WK perpendicular to RN

ons.); the angles at K and N are right angles, at S are equal; hence triangles KWS, NMS 26), and ∴ $SK = SN$; but RS is double of SK is therefore double of SN.

EXERCISE 34.

A, B, C, D are four points taken in order on the circumference of a circle; the straight lines AB, CD produced intersect at P, and AD, BC at Q: show that the straight lines QM, PM which respectively bisect the angles AQC, APC contain a right angle at M, their point of intersection.

(Fig. 34, Plate III.)—$AQC + APC + 2NCQ =$ the angles ADC, ABC = two right angles (**I. 32** and **III. 22**); ∴ their halves, or the angles MQC, MPC, NCQ equal one right angle; but $MNP = MQC + NCQ$, hence $MNP + MPC =$ one right angle, and the angle QMP is ∴ a right angle.

EXERCISE 35.

ACB, APB are two equal circles, the centre of APB being in the circumference of ACB, AB being the common chord; if any chord AC, of ACB be produced to cut APB in P, the triangle PBC is equilateral.

(Fig. 35, Plate III.)—$BC = BP$ (**III. 26, 29**); and the angle ACB is double of BPC, because it is equal to the angle which would be formed if C were at the centre of APB (**III. 21, 20**); but it is also equal to $CBP + CPB$ (**I. 32**), ∴ $CBP = CPB$, whence the sides CB and CP are equal, and ∴ PBC is an equilateral triangle.

EXERCISE 36.

If two circles cut each other so that the circumference of one passes through the centre of the other, and from either point of intersection a straight line be drawn cutting both circumferences: the part intercepted between the two circumferences will be equal to the chord drawn from the other point of intersection to the point where the former line meets the inner circumference.

(Fig. 36, Plate III.)—Let KMN be a circle, and KCN another passing through its centre C; let KOM be a line drawn cutting both circumferences; join ON: then $OM = ON$.

For, by joining CK, CN, we have the angle KCN = angle KON (III. 21), but KCN is double of KMN (III. 20), ∴ KON is double of KMN; but since $KON = KMN + ONM$ (I. 32), ∴ $KMN = ONM$, and the sides OM, ON are equal (I. 6). If RKS be a tangent to KCN, meeting the concave circumference of KMN at S, and KN be joined; $KN = KS$. For angle RKN = angle KCN (III. 32) = twice KSN; ∴ angle KSN = angle KNS (I. 32), whence $KN = KS$.

EXERCISE 37.

Draw two tangents to a given circle, which shall contain an angle equal to a given rectilineal angle.

(Fig. 37, Plate III.)—Find M, the centre of the circle (III. 1), and draw the radii MN, MK containing an angle equal to the supplement of the given angle; through the points K, N draw the tangents KR, NR meeting in R: then RK, RN are the tangents required.

The angles at K and N are right angles, ∴ the sum of the angles KRN, KMN equals two right angles, and hence the angle at R equals the given angle.

EXERCISE 38.

If from any point without a circle, lines be drawn cutting the circle and making equal angles with the longest line, they will cut off equal segments.

(Fig. 38, Plate III.)—Draw ML through the centre of the given circle from the given point M, and draw MKS, MRN, making equal angles with ML, and cutting the circle in K, S and R, N respectively: the segment KQS = the segment RWN.

Join KR.

KMR is an isosceles triangle (III. 8), ∴ the angles SKR, NRK are equal; ∴ the arc RNS = the arc KSN; from each take the common part SN, and the remainders RN, KS are equal, and hence the segments KQS, RWN are equal.

EXERCISE 39.

If from any point in the arc of a segment of a circle a line be drawn perpendicular to the base, and from the greater segment of *the base and the greater arc parts be cut off respectively equal to the less: the remaining part of the base shall be equal to the chord of the remaining arc.*

(Fig. 39, Plate III.)—From any point M in the arc DMR, draw MS at right angles to DR; make $SL = SR$, and the arc MK = the arc MR; join KD: $KD = DL$.

Join MR, ML, MK, and KL.

$ML = MR$ (I. 4) $= MK$ (III. 29), and because $DKMR$ is a quadrilateral inscribed in a circle, $\therefore$ $DKM + DRM$ = two right angles, or, $DKM + MLR$ = two right angles, $\therefore$ (I. 13) $DKM = DLM$; but since $ML = MK$ the angles MKL, MLK are equal (I. 5); $\therefore$ the angles DKL, DLK are equal, and the sides KD, DL are equal.

EXERCISE 40.

If from any point outside a circle tangents be drawn; the angle which they include is double the angle between the chord joining their point of contact and a radius drawn from one of them.

(Fig. 40, Plate III.)—Let MS, MR be the tangents; SR the line joining their points of contact, and SK the radius: angle SMR = twice angle KSR.

Join KR.

The angles at S and R are right (III. 18), $\therefore$ $SMRK$ is inscribable in a circle; and because the arc SKR is double the arc KR, $\therefore$ the angle SMR is double of KSR (III. 27).

EXERCISE 41.

A chord POQ cuts the diameter AB of a circle in O, at an angle equal to half a right angle: $PO^2 + OQ^2 = 2MB^2$, M being the centre.

(Fig. 41, Plate III.)—Draw MN perpendicular to PQ, and $\therefore$ (III. 3) bisecting it: $PO^2 + OQ^2 = 2PN^2 + 2NO^2$ (II. 9) $= 2PO \cdot OQ + 4NO^2$ (II. 5); but $2PO \cdot OQ = 2AO \cdot OB$ (III. 35) and since the angle at N is right and the angle at O half a right angle, $\therefore$ the angle at M = half a right angle, and $NO = NM$ (I. 6), and hence $4NO^2 = 2MO^2$; $\therefore$ $PO^2 + OQ^2 = 2AO \cdot OB + 2MO^2 = 2MB^2$ (II. 5).

EXERCISE 42.

Through a given point P *outside a circle* MKR, *draw a straight line, such that the part intercepted by the circle shall have a given length.*

(Fig. 42, Plate III.)—With any point L as centre, and the given length as radius, describe an arc cutting the circle in R, and with the centre of the given circle as centre, describe a circle touching LR; from P draw the tangent $PMSK$ (III. 17), cutting the given circle in M and K; PK is the line required.

$MK = LR$ (III. 14).

If P be within the given circle the same construction holds.

EXERCISE 43.

If two circles touch each other, any straight line drawn through the point of contact will cut off similar segments.

(Fig. 43, Plate III.)—Let APE and BPF be two circles touching each other in the point P: then the segment AGP is similar to the segment BHP, and the segment AEP to the segment BFP.

Through P draw a common tangent CD.

Because the angle APC = angle BPD (I. 15), and the angle APC = any angle in the segment AEP (III. 32), and the angle BPD = any angle in the segment BFP, therefore those segments are similar (III. *Def.* 11); hence the remaining segments of these circles are similar.

If one touch the other internally it can be proved in the same manner.

EXERCISE 44.

Two circles intersect at the points A *and* B, *from which are drawn chords to a point* C *in one of the circumferences, and these chords, produced if necessary, cut the other circumference at* D *and* E: *show that the straight line* DE *cuts at right angles that diameter of the circle* ABC *which passes through* C.

(Fig. 44, Plate III).—Join AB; draw $CKGF$ through the centre K to cut DE in F; and join BG.

Because $ABED$ is a quadrilateral in the circle ABE, the angles BAD and BED are together equal to two right angles (III. 22); but BAD and BAC are also together equal to two right angles (I. 13), $\therefore BAC = BED$ (*Ax.* 3); and, as $BAC = BGC$

(III. 21), *BGC* = *BED* (*Ax.* 1). Now, in the two triangles *CEF* and *CBG*, the angle *CEF* = *CGB*, and the angle *FCE* is common to both, ∴ the remaining angles *CFE* and *CBG* are equal (I. 32); but the angle *CBG* is a right angle (III. 31), ∴ the angle *CFE* is a right angle.

EXERCISE 45.

The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, intersect at the same point.

(**Fig. 45, Plate III.**)—Let *ACDEB* be a segment of a circle, *AB* its chord, and *ACB*, *ADB*, and *AEB* triangles on the same base *AB* and having (III. 21) equal vertical angles: the straight lines bisecting these angles must all meet in the same point.

Bisect the arc *APB* in *P* (III. 30), and draw *PC*, *PD*, and *PE*.

Because *AP* = *PB*, the angles *ACP*, *PCB*, *ADP*, *PDB*, *AEP*, and *PEB* are all equal (III. 27), ∴ the lines *CP*, *DP*, and *EP*, bisecting the angles at *C*, *D*, and *E* all meet in the same point *P*.

EXERCISE 46.

If a quadrilateral be inscribed in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

(**Fig. 46, Plate IV.**)—Let *ABCD* be a quadrilateral inscribed in a circle, and let the line *EKF* be drawn making equal angles, *FED* and *EKC* with *AD* and *BC*: it makes equal angles, *AHE* and *DFE*, with *AB* and *DC* produced.

Because the angle *BKH* = angle *EKC* (I. 15), and angle *FED* = angle *EKC* (*Hyp.*), the angle *BKH* = angle *FED* (*Ax.* 1); and as the angles *ABC* and *HBK* = two right angles, and the angles *ABC* and *ADC* = two right angles (III. 22), ∴ the angle *HBK* = angle *ADC*; hence in the two triangles *HBK* and *FED*, the angle *HBK* = angle *FDE*, and the angle *BKH* = angle *FED*, the remaining angles *BHK* and *EFD* are equal (I. 32), that is, the line *EF* makes equal angles with *AB* and *DC* produced.

EXERCISE 47.

If a quadrilateral, having two of its opposite sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to the parallel sides, and terminated by the other two sides, shall be equal to a fourth part of the perimeter of the figure.

(Fig. 47, Plate IV.)—Let $ABCD$ be a quadrilateral, described about a circle, having its two sides AD and BC parallel, and let the straight line EF be drawn parallel to AD and BC through the centre K; then this straight line, EF, is one-fourth of the perimeter of the quadrilateral $ABCD$.

Draw DB, cutting EF in G; join KM and KN.

Because AD and BC are parallel tangents at the points M and N, the line MN is a diameter; and, since $MK = KN$, $AE = EB$, and $DF = FC$, $\therefore$ (*Ex.* 147, Part I.) $EG =$ half AD, and $GF =$ half BC, that is (*Ax.* 1) $EF =$ half the sum of AD and BC; but $AD + BC = AB + DC$; $\therefore$ $EF =$ one-fourth the sum of AD, BC, AB, $DC =$ one-fourth of the perimeter of the quadrilateral.

EXERCISE 48.

To draw a straight line to touch each of two given circles.

(Fig. 48, Plate IV.)—Let BMN and DGS be the two circles.

From C, the centre of the larger circle, as centre, and radius equal to the difference of the radii of the two circles describe the circle APW; from K, the centre of the less circle, draw KA a tangent to the circle APW; join CA, and produce it to meet the circumference of BMN in B; from K draw KD parallel to AB and join B and D; BD is a common tangent to BMN and DGS.

Because AB is equal and parallel to KD, $\therefore$ BD is equal and parallel to AK (I. 33); and as the angles at A are right angles, the angles at B and D are right angles (I. 29 and 34), $\therefore$ BD is a common tangent to both circles.

EXERCISE 49.

To draw that diameter of a given circle which shall pass at a given distance from a given point.

(Fig. 49, Plate IV.)—Let *BHDF* be the given circle, *P* the given point, and *PA* equal the given distance.

From *P* as centre, and *AP* as radius, describe a circle; from *C* the centre of the circle *BHDF* draw *CB* a tangent to it (III. 17), meeting it in *A*, and produce *BC* to the circumference in *D*: *BD* is the diameter required.

Because *BD* is a tangent to the circle *AMGN* at the point *A*, and *AP* is a radius, *AP* is perpendicular to *BD* (III. 18); and *BD* is the diameter required.

If the given point be without the circle, the same method of solution can be employed.

EXERCISE 50.

A and B are the centres of two circles which touch internally at C, and also touch a third circle, whose centre is D, externally and internally respectively at E and F: show that the angle ADB is double of the angle ECF.

(Fig. 50, Plate IV.)—Produce *CA* to *G*.

Because the angle *GAF* is equal to the two angles *ACF* and *AFC* (I. 32), and that these two last are equal (I. 5), the angle *GAF* is double of the angle *ACF*. But the angle *GAF* = angles *ABD* and *ADB* (I. 32), $\therefore ABD + ADB = 2ACF$ (*Ax.* 1). Again, angle $ABD = BCE + BEC = 2BCE$, $\therefore$ angle $ADB = 2ECF$ (I. 32, *Cor.*).

Note.—As the angle *ADB* is also double of the angle *AFE*, it follows that the angle *ECF* = angle *AFE* (*Ax.* 1), or *DEF*.

EXERCISE 51.

If two equal circles be placed at such a distance apart that the tangent drawn to either of them from the centre of the other is equal to a diameter, show that they will have a common tangent equal to the radius.

(Fig. 51, Plate IV.)—Let *AEH* and *BDF* be two equal circles, and let them be so placed that the tangent *CA* from the centre *C* to the circle *AEH* may be equal to a diameter; then they have a common tangent equal to the radius.

From C draw the radius CB parallel to AK; join KB; join also DE: then DE is a common tangent, and is equal to BC or KA.

The figure $ACBK$ is evidently a rectangle; and as $CD = DA$ (*Hyp.*), and $KE = EB$, each of the figures $ADEK$ and $DCBE$ is a square, and $ED = BC = AK$; and it is also a common tangent since the angles at D and E are right angles.

EXERCISE 52.

If from the centre of a circle a line be drawn to any point in the chord of an arc, the square on that line, together with the rectangle of the segments of the chord, will be equal to the square on the radius.

(Fig. 52, Plate IV.)—Let AB be a chord of the circle AHB, C the centre, E a point in AB; then the rectangle $BE \cdot EA$, together with the square on EC, is equal to the square on the radius.

From C let fall the perpendicular CD, and join AC and EC.

The rectangle $BE \cdot EA + ED^2 = AD^2$ (II. 5). To each of these equals add CD^2, and the rectangle $BE \cdot EA + ED^2 + DC^2 = AD^2 + DC^2 = AC^2$; and as $ED^2 + DC^2 = EC^2$, $\therefore$ the rectangle $BE \cdot EA + EC^2 = AC^2$.

EXERCISE 53.

If two circles intersect, the common chord produced bisects the common tangent.

(Fig. 53, Plate IV.)—Let RK be the common chord of the two circles RKM, RKN, which on being produced meets the common tangent MN at L: MN is bisected in the point L.

$LN^2 = RL \cdot LK = LM^2$ (III. 36); $\therefore LN = LM$.

EXERCISE 54.

AB is the diameter of a semicircle, D and E are any two points in its circumference: show that if the chords joining A and B with D and E each way intersect at F and G, then FG produced cuts AB at right angles.

(Fig. 54, Plate IV.)—Let FG cut AB at K, and join DE.

$BAE = BDE$ (III. 26); and because the angles BDF, *AEF* are right, ∴ *GDFE* is inscribable in a circle; wherefore the angles *EDG*, *EFG* are equal (III. 26); hence the angles *BAE*, *BFK* are equal, and the angle *ABE* is common to the triangles *ABE*, *FBK*; ∴ the remaining angles *AEB*, *FKB* are equal; but *AEB* is a right angle, wherefore the angle *FKB* is also right.

EXERCISE 55.

If two circles cut each other, the tangents drawn to them from any point in the common chord produced are equal.

(Fig. 55, Plate IV.)—Let *ABD* and *ABE* be two circles cutting each other at the points *A* and *B*; and let *CD* and *CE* be tangents to them from *C* any point in the common chord *AB*: then *CD* is equal to *CE*.

CD^2 = the rectangle $AC \cdot CB$, and CE^2 = the rectangle $AC \cdot CB$ (III. 36), ∴ $CD^2 = CE^2$, and $CD = CE$.

EXERCISE 56.

AB is a common chord of two circles; through C any point of one circumference straight lines CAD, CBE are drawn terminated by the other circumference: show that the arc DE is invariable.

(Fig. 56, Plate IV.)—Join *AE*.

The angles at *C* and *E* are invariable (III. 21), ∴ their sum, or the angle *DAE*, is also invariable whence the arc *DE* is invariable (III. 26).

EXERCISE 57.

Two tangents AB, AC are drawn to a circle; D is any point on the circumference outside of the triangle ABC: show that the sum of the angles ABD, ACD is constant.

(Fig. 57, Plate IV.)—The sum of angles *BAC*, *BDC* is invariable; hence the sum of angles *ABD*, *ACD* is constant, the whole four being equal to four right angles.

EXERCISE 58.

If two chords intersect within a circle, the angle which they include is measured by half the sum of the intercepted arcs.

(Fig. 58, Plate IV.)—Let the chords *AB*, *CD* of the

PLATE III.

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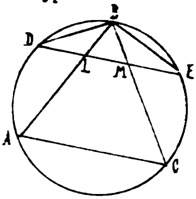

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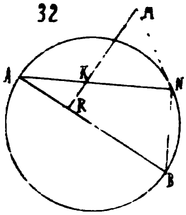

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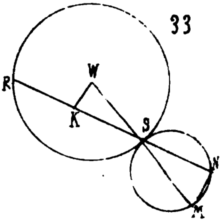

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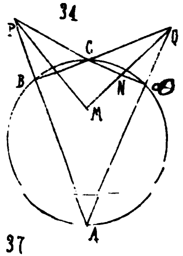

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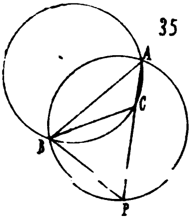

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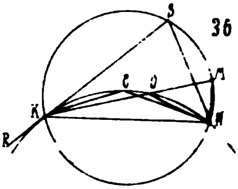

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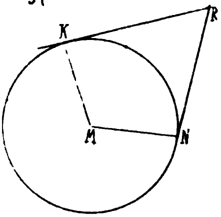

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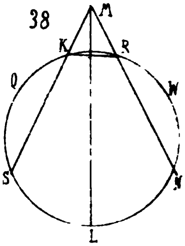

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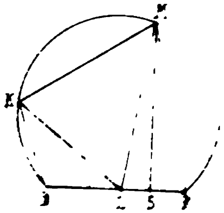

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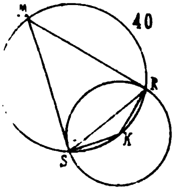

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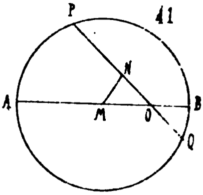

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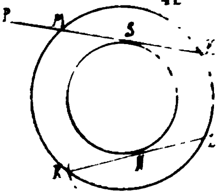

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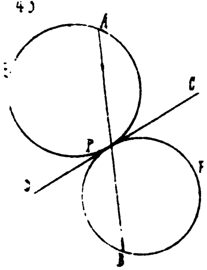

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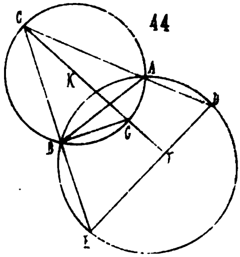

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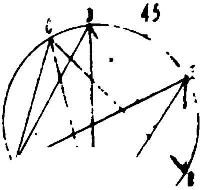

PLATE IV

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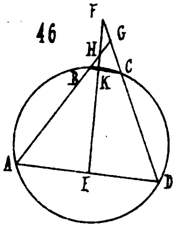

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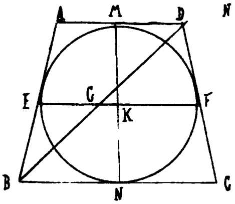

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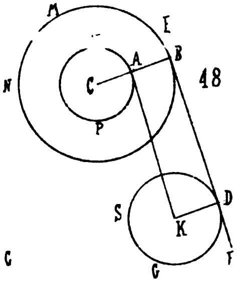

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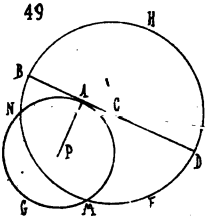

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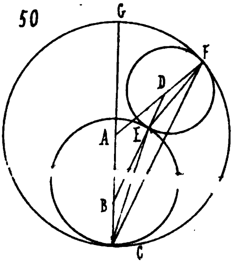

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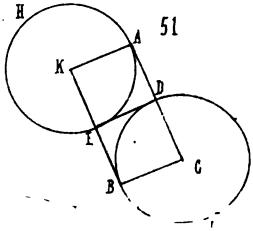

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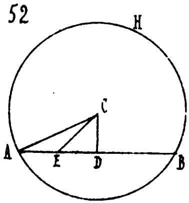

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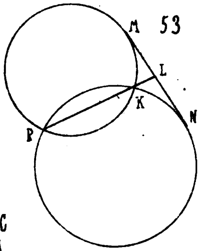

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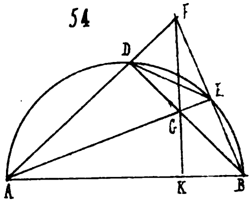

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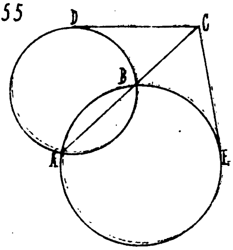

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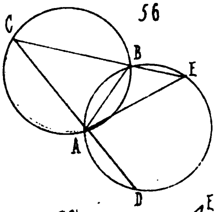

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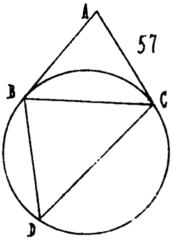

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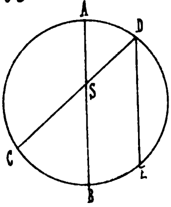

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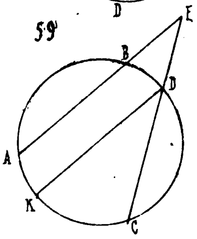

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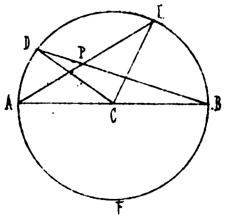

circle *ADBC* intersect at *S*: the angle *CSB* is measured by half the sum of the arcs *AD*, *CB*.

Draw *DE* parallel to *AB*, and meeting the circumference at *E*.

CDE is measured by half the sum of the arcs *CB*, *BE* (III. 20), but since *DE* is parallel to *AB*, the arcs *BE*, *AD* are equal, and angle *CSB* = *CDE*; ∴ angle *CSB* is measured by half the sum of the arcs *AD*, *CB*.

EXERCISE 59.

If two chords produced intersect without a circle, the angle which they include is measured by half the difference of the intercepted arcs.

(Fig. 59, Plate IV.)—Let the chords *AB*, *CD*, being produced, intersect at *E*: angle *AEC* is measured by half the difference of the arcs *AC*, *BD*.

Draw *DK* parallel to *AB*, meeting the circumference at *K*.

Because *KD* and *AB* are parallel, the arcs *AK*, *BD* are equal, ∴ the arc *KC* is the difference between the arcs *AC*, *BD*, and the angles *KDC*, *AEC* are equal, but *KDC* is measured by half the arc *KC* (III. 20), hence *AEC* is also measured by half the arc *KC*, that is, by half the difference of the arcs *AC*, *BD*.

EXERCISE 60.

AB is the diameter of a circle whose centre is C, and DCE is a sector having the arc DE constant; join AE and BD intersecting at P: show that the angle APB is constant.

(Fig. 60, Plate IV.)—Because the arc *DE* is constant, and the arc *AFB* is constant; and as the angle *APB* is measured by half the sum of these arcs (*Ex.* 58), it must also be constant.

EXERCISE 61.

In a circle two chords of given length are drawn so as not to intersect, and one of them is fixed in position; the opposite extremities of the chords are joined by straight lines intersecting within the circle: show that the locus of the point of intersection will be a portion of the circumference of a circle, passing through the extremities of the fixed chord.

(**Fig. 61, Plate V.**)—Let *AB* and *CD* be two chords of given length in the circle *ACDB*; and let *AB* be fixed; join *B*, *C* and *A*, *D*, and let *BC* and *AD* intersect at *E*: then the angle *AEB* is invariable, whatever may be the position of *CD* on the same side of *AB*.

Because the angle *AEB* is measured by half the sum of the arcs subtended by *AB* and *CD*, it is invariable, whatever may be the position of *CD*; and hence it will always be found in the same segment of a circle (III. 21)—namely, that having *AB* for its chord, and hence passing through *A* and *B*.

EXERCISE 62.

APB is a fixed chord passing through P a point of intersection of two circles AQP, BPR; and QPR is any other chord of the circles passing through P: show that AQ and RB when produced meet at a constant angle.

(**Fig. 62, Plate V.**)—Let *AQ* and *RB* meet at *M*.

AQP is constant (III. 21); ∴ *MQR* is constant; for the same reason *BRP* is constant; whence in the triangle *QMR* the sum of the angles *MQR*, *MRQ* is constant, and ∴ *QMR* is also constant.

EXERCISE 63.

Two segments of circles are on the same base AB, and P is any point in the circumference of one of the segments; the straight lines APD and BPC are drawn meeting the circumference of the other segment at D and C; AC and BD are drawn intersecting at Q: show that the angle AQB is constant.

(**Fig. 63, Plate V.**)—Because the angles *ACB* and *ADB* are equal (III. 21) and invariable, their supplements *PCQ* and *PDQ* are equal and invariable (I. 13); the angle *APB* is also invariable, and, hence (I. 15), *CPD* is invariable, ∴ the angle *CQD* (or *AQB*) must be constant, since these four angles are equal to four right angles.

EXERCISE 64.

AB, BC, CD are three adjacent sides of any polygon inscribed in a circle; the arcs AB, BC, CD are bisected at L, M, N; and LM cuts BA, BC respectively at P and Q: show that BPQ is an *isosceles triangle; and that the angles ABC, BCD are together double of the angle LMN.*

(Fig. 64, Plate V.)—*BPQ* is measured by half the sum of the arcs *AL*, *BM*; and *BQP* is measured by half the sum of the arcs *LB*, *MC*, but $AL + BM = LB + MC$, and therefore $BPQ = BQP$, hence *BPQ* is isosceles.

Again, *ABC* stands on the arcs *AD* + twice *DN*, and *BCD* on *AD* + twice *AL*, but *LMN* stands on the arcs *AD*, *AL*, *DN*, which are equal to half the sum of the arcs that *ABC* and *BCD* stand on, and therefore the sum of the angles *ABC*, *BCD* is double of the angle *LMN*.

EXERCISE 65.

On a straight line AB as base, and on the same side of it are described two segments of circles; P is any point in the circumference of one of the segments, and the straight line BP cuts the circumference of the other segment at Q: show that the angle PAQ is equal to the angle between the tangents at A.

(Fig. 65, Plate V.)—Because *AQ* is a chord of the inner segment, *AQB*, and *AM* is a tangent, the angle *QAM* = the angle *QBA* in the alternate segment (III. 32). For the same reason the angle $NAP = PBQ$; and ∴ the angle *QAM* = angle *NAP* (*Ax.* 1). From these equal angles take the common angle *MAP*, and the remaining angle *PAQ* = the remaining angle *NAM* (*Ax.* 3); that is, *PAQ* = the angle between the tangents *AM* and *AN*.

EXERCISE 66.

In a given circle to inscribe a rectangle equal to a given rectilineal figure.

(Fig. 66, Plate V.)—Let *AFBE* be the given circle.

Draw any diameter *AB*, and to it apply a rectangle equal to the given rectilineal figure (I. 45); let *ADCB* be that rectangle; let *DC* meet the circumference in *E*; join *AE*, *BE*; and draw *AF*, *FB* equal to *AE*, *EB*; then *AFBE* is the required rectangle.

The rectangle *ADCB* = twice triangle *AEB* (I. 41), and the rectangle *AFBE* = twice triangle *AEB*, ∴ rectangle *ADCB* = rectangle *AFBE*. But the rectangle *ADCB* = the given rectilineal figure (*Cons.*), ∴ the rectangle *AFBE* = the given rectilineal figure (*Ax.* 1).

EXERCISE 67.

A quadrilateral is bounded by the diameter of a circle, the tangents at its extremities, and a third tangent: show that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

(Fig. 67, Plate V.)—Let $ABDE$ be the quadrilateral bounded by the diameter AB, the two tangents AE and BD at the extremities of AB, and a third tangent ED: then the area of $ABDE$ is equal to half the rectangle contained by AB and ED. Let C be centre.

Join CE, CF, and CD.

The triangle FCD = triangle BCD (I. 47 *Cor.*), and the triangle FCE = triangle ACE; but the area of the triangle FCD = half FC multiplied by FD (I. 41), ∴ the area of the quadrilateral $FCBD$ = FC multiplied by FD. For the same reason the area of the quadrilateral $FCAE$ = FC multiplied by FE; and ∴ the total area of these two quadrilaterals = FC multiplied by $FD + FE$ = FC multiplied by ED = half AB multiplied by ED.

EXERCISE 68.

A triangle ABC is inscribed in a circle: show that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

(Fig. 68, Plate V.)—The angles in the segment of which AB is the chord, equals the sum of the angles on the arc ACB, *i.e.*, those at A and B; for a similar reason those in the segments of which CB, CA are the chords equal the sum of the angles at B and C, A and C respectively, whence the sum of the angles in the three segments equals twice the angles at A, B, C, namely four right angles.

Note.—This, perhaps, may be seen more clearly by taking any point in each of the arcs and joining each point with the angular points of the triangles.

EXERCISE 69.

A quadrilateral is inscribed in a circle: show that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

(Fig. 69, Plate V.)—Let *ABCD* be a quadrilateral inscribed in a circle, and let *AEB*, *BFC*, *CGD*, and *DHA* be angles in the four segments exterior to the quadrilateral: their sum is equal to six right angles.

Join *BD*.

The angles *BDC* and *BFC* are together equal to two right angles (III. 22); the angles *ADB* and *AEB* = two right angles; the angles *ABD* and *AHD* = two right angles; and the angles *DBC* and *DGC* = two right angles; ∴ the two angles *ABC* and *ADC* together with the four angles in the exterior segments are equal to eight right angles; but *ABC* and *ADC* are together equal to two right angles, hence the four angles in the exterior segments are together equal to six right angles.

EXERCISE 70.

The straight lines bisecting any angle of a quadrilateral inscribed in a circle and the opposite exterior angle meet in the circumference of the circle.

(Fig. 70, Plate V.)—Let *AN* bisect the angle *A* of the quadrilateral *ABCD*; join *NC*: the line *NC* bisects the opposite exterior angle *DCM*.

Because *ABCD* is inscribed in a circle, the angle *DAB* = the angle *DCM*.

Again, *NAD* = *DCN* (III. 27), but *NAD* = half of *DAB*, ∴ *DCN* is also half of *DCM*.

EXERCISE 71.

AB is the diameter of a semicircle, P is a point on the circumference, PM is perpendicular to AB; on AM, BM as diameters two semicircles are described, and AP, BP meet these latter circumferences at Q, R: show that QR will be a common tangent to them.

(Fig. 71, Plate V.)—Bisect *AM*, *BM* in *L*, *K* respectively, and join *QL*, *QM*, *RK*, *RM*.

QPRM is a rectangle, because the angles at *Q*, *P*, *R* are right angles being in semicircles, and ∴ its diagonals are equal and bisect each other, whence the angles *RQM*, *PMQ* are equal, and since *LM* = *LQ* the angles *LQM*, *LMQ* are also equal; ∴ the whole angle *LQR* = the whole angle *LMP*, and is ∴ a right angle, whence *RQ* is a tangent to the semicircle *AQM*; in like manner it may be shown to be a tangent to the other semicircle.

EXERCISE 72.

If ABC be a triangle, and BE, CF the perpendiculars from the angles on the opposite sides, and K the middle point of the third side: show that the angles FEK, EFK are each equal to A.

(Fig. 72, Plate V.)—Because CEB, CFB are right angles, CB is the diameter of a circle passing through E and F; $\therefore$ KE, KF are radii, hence the angles KFE, KEF are equal.

Again, because $CEFB$ is a quadrilateral inscribable in a circle, the angles AFE, KCE are equal, but $KEC = KCE = AFE$; and since the angles of the triangle EFA equal two right angles or the angles at E, and $KEC = AFE$, the whole angle $KEA = FEA + FAE$; from these take the common angle FEA and the remainders KEF, FAE are equal.

EXERCISE 73.

If any two circles touch each other in the point M, and lines be drawn through M at right angles to each other, one cutting the circles in N, R, and the other cutting them in L, H: the sum of the squares of LH, NR = the square of KMO, the sum of the diameters.

(Fig. 73, Plate V.)—Join KL, OR, and produce them to meet at Z: draw KN, OH.

$KNRZ$, $HOZL$ are rectangles, $\therefore$ $NR^2 + HL^2 = KZ^2 + OZ^2 = KO^2$.

EXERCISE 74.

Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

(Fig. 74, Plate V.)—Draw any tangent ANB touching the circle at N; from N draw the chord NM making the angle ANM equal one-third of two right angles; NM divides the circle in the manner required (III. 32).

EXERCISE 75.

C is the centre of a given circle, CA a straight line less than the radius: find the point of the circumference at which CA subtends the greatest angle.

(Fig. 75, Plate V.)—From A draw AP perpendicular to CA meeting the circumference at P: then P is the point required.

Join CP, and describe the circle CPA having CP as diameter; take any other point M in the outer circumference, and join CM, AM, the former cutting the inner circumference at N; join AN.

The angle CPA = angle CNA (III. 21); but CNA is greater than CMA (I. 16), and ∴ CPA is also greater than CMA, whence P is the point required.

EXERCISE 76.

Describe a circle which shall pass through a given point and touch a given straight line at a given point.

(**Fig. 76, Plate VI.**)—Let M be the point in the given line AB, and N the other point.

Join MN; and draw MR perpendicular to AB; draw NS cutting MR in S, making the angle $SNM = SMN$: S is the centre of the required circle.

If a circle be described with S as centre, and SM as radius, it will touch AB in the point M, because AMS is a right angle.

It will also pass through the point N because $SM = SN$ (I. 6).

EXERCISE 77.

Describe a circle which shall touch a given circle at a given point, and touch a given straight line.

(**Fig. 77, Plate VI.**)—Draw a diameter MD from the given point D in the given circle MLD, and produce; draw the tangent DC cutting the given line, SR, in C; on SR take the part $CB = CD$, and draw BA at right angles to SR cutting MD produced in A: A is the centre of the circle required.

Join CA.

Then $AB = AD$ (I. 47), and the angles at D and B are right angles; ∴ A is the centre of the circle required.

EXERCISE 78.

With a given radius X, to describe a circle touching two given circles.

(**Fig. 78, Plate VI.**)—Produce the radii AK, BN of the given circles to O and W respectively, making each of the produced parts equal to X, and with these produced lines (AO, BW) as radii describe two arcs cutting in M; join AM, BM, and let them cut the circles in the points L and Q respectively; then since $ML = MQ$, ∴ M is the centre of the circle required.

EXERCISE 79.

A circle is drawn to touch a given circle and a given straight line: show that the points of contact are always in the same straight line with a fixed point in the circumference of the given circle.

(Fig. 79, Plate VI.)—Let the circle *BMF* touch the given circle *ABH* in the point *B*, and the straight line *DE* in the point *F*; the points *B* and *F* are in the same straight line with the fixed point *A*, the extremity of the perpendicular on *DE* from the centre *C*.

Because *AD* is parallel to *KF*, and *CK* meets them, the angle *ACB* = angle *BKF* (I. 29); and because *AC* = *CB*, the angle *CAB* = angle *CBA* (I. 5); for the same reason the angle *KBF* = angle *KFB*; but since angle *ACB* = angle *BKF*, the sum of angles *CAB* and *CBA* = sum of angles *KBF* and *KFB* (I. 32), and their halves, the angles *CBA* and *KBF* are equal (*Ax.* 7), and therefore *AB* and *BF* are in the same straight line.

EXERCISE 80.

To describe a circle that shall have a given radius and touch a given circle and a given straight line.

(Fig. 80, Plate VI.)—Let *AMN* be the given circle, *DE* the given straight line, and *FP* equal to the given radius.

From *C* the centre of the given circle draw *CP* perpendicular to *DE*; cut off *FP* equal to the given radius; from *C* as centre, with distance equal to the sum of the two radii—viz., of *CS* + *SG* = *CR* + *PF*, describe an arc cutting *FG*, a parallel to *DE*, in *G*: the point *G* is the centre of the required circle.

From *G* draw *GH* parallel to *PF*, or perpendicular to *DE*; then *GH* = *PF* (I. 34), and *PF* = *SG* (*Cons.*), ∴ *GH* = *SG* (*Ax.* 1); and the circle *HSW* touches the given circle *AMN*, the given straight line *DE*, and has a radius equal to the given straight line *PF*.

EXERCISE 81.

To describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

(Fig. 81, Plate VI.)—Let *ABE* be the given circle, *DH* the given straight line, and *P* the given point in the given straight line; it is required to describe a circle which shall touch *ABE*, pass through the point *P*, and have its centre in the line *DH*.

Through *C*, the centre of the given circle, draw *AE* parallel to *DH*; join *A* and *P*, *C* and *F*, the point in which *AP* cuts the circumference of *ABE*, and produce *CF* to meet *DH* in *K*; then *K* is the centre of the required circle.

Because *AE* is parallel to *DH*, and *AP* meets them, the angle *CAF* = angle *KPF* (I. 29); the angle *CAF* = angle *CFA* (I. 5); the angle *CFA* = angle *KFP* (I. 15); ∴ the angle *KFP* = angle *KPF* (*Ax.* 1), and hence *KF* = *KP* (I. 6), and the circle described from *K* as centre, with *KF* as radius, will touch the circle *ABE* in *F* (III. 12), and pass through the given point *P* in the given straight line *DH*.

EXERCISE 82.

To draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle.

(Fig. 82, Plate VI.)—Let *DEF* and *MGH* be two circles. Take any point *G* in the circumference of *MGH*, and with *G* as centre and distance equal to the given straight line, describe an arc cutting the circumference in *H*; join *GH*; from the centre *K* let fall the perpendicular *KW* on *GH*; from the centre *K* and distance *KW* describe a circle; and draw *AN* a common tangent to both the circles *DEF* and *BSW* cutting the circumference of *MGH* in *M*, *N*. The part *MN* is equal to *GH* = the given straight line (III. 14).

EXERCISE 83.

To describe a circle touching a given straight line at a given point, such that tangents drawn to it from two given points in the straight line may be parallel.

(Fig. 83, Plate VI.)—Let *AB* be the given straight line, *P* the given point in which the circle must touch the given line *AB*, and *A* and *B* the two given points from which parallel tangents are to be drawn to the required circle.

On *AB* describe the semicircle *ACB*; draw *PC* at right angles to *AB*; join *AC*, *BC*; at the point *A*, with the line *AC*, make the angle *CAF* equal to the angle *CAP* (I. 23), and from *C* draw *CD* perpendicular to *AF*; at *B*, with the line *BC*, make the angle *CBG* equal to the angle *CBP*, and from *C* draw *CE* perpendicular to *BG*; then *CP*, *CD*, and *CE* are all equal, and therefore (III. 9) *C* is the centre of the required circle.

It is evident from the construction that *CD*, *CP*, and *CE* must be equal (I. 26); and since the angles *CAP* and *CBP* are together equal to one right angle; the angles *DAP* and *EBP* are equal to two right angles, and ∴ *AD* is parallel to *BE*.

Note.—This may also be readily effected by drawing a tangent to the circle *ACB* through the point *C*, and then drawing *AD* and *BE* perpendicular to the tangent, as in the following problem.

EXERCISE 84.

To describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

(Fig. 84, Plate VI.)—Let *AB* be the given straight line, and *AD* equal the given radius; it is required to describe a circle with radius equal to *AD*, and touching the given line *AB*, such that tangents to the required circle from *A* and *B* may be parallel.

From *A* draw *AD* at right angles to *AB*, and equal to the given radius; on *AB* describe a semicircle; from *D* draw *DC* parallel to *AB*; through the point *C* draw a tangent to the semicircle *ACB*; from *C* draw *CE* perpendicular to *AB*, and from *A* and *B* draw *AF* and *BG* perpendicular to the tangent *FCG*: then *C* is the centre of the circle, for *CF*, *CE*, and *CG* are equal.

Join *AC* and *BC*.

Because *CG* is a tangent, and *CB* a chord, the angle *GCB* = angle *CAE* (III. 32); the angles at *E* and *G* are right angles (*Cons.*); and *BC* is common to both triangles—viz., *BCE* and *BCG*, ∴ *CG* = *CE* (I. 26). Again, since *CF* is a tangent, and *AC* a chord, the angle *ACF* = angle *ABC*; and because the angle *ACB*, in a semicircle, is a right angle and *CE* a perpendicular, the angle *ACE* = angle *ABC*, ∴ the angle *FCA* = angle *ACE*, and the angles at *E* and *F* are right angles, and *AC* common to both triangles—viz., *ACF* and *ACE*; and ∴ *CF* = *CE* (I. 26), and the three perpendiculars *CG*, *CE*, and *CF* are equal, and *C* is the centre of the circle (III. 19).

Again, since the angles at *F* and *G* are right angles, the tangents, *AF* and *BG*, to the circle described from *C* as centre, and *CG*, *CE*, or *CF*, as radius, are parallel (I. 28).

EXERCISE 85.

To describe a circle which shall pass through a given point and touch two given straight lines.

(Fig. 85, Plate VI.)—Let *AB* and *AF* be the two straight lines, and first let them not be parallel; and let *G* be the given point.

Draw *AN* bisecting the angle *BAF*; through *G* draw *MGHK* at right angles to *AN*, and make *PH* equal to *PG*; from *AM* cut off a part *ME*, so that the rectangle $HM \cdot MG$ may be equal to the square on *ME* (II. 14); and through *E*, *G*, and *H* describe the circle *DHGE*, which is the circle required. From *E* draw *EC* at right angles to *AM* and *CD* at right angles to *AB*.

Because the angles at *A* are equal (*Cons.*), the angles at *D* and *E* right angles, and *AC* common, the sides *CE* and *CD* are equal (I. 26); and ∴ the circle described from *C* as centre and radius equal to *CE*, or *CD* touches *AB* and *AF*. It also passes through *G* and *H*, because the rectangle $HM \cdot MG = GK \cdot KH = ME^2 = KD^2$ (III. 37). If the lines be parallel, bisect the common perpendicular, as in *P*, and make *PH* equal to *PG*; then find the side *ME* of a square equal to the rectangle $HM \cdot MG$.

EXERCISE 86.

To describe a circle which shall touch two given circles and a given straight line.

(Fig. 86, Plate VI.)—Let *GHM* and *ENW* be the two given circles, and *AB* the given straight line; it is required to describe a circle touching the two circles *GHM*, *ENW*, and *AB*.

From *K* the centre of the larger circle, if they be unequal, describe a circle with the difference of the radii of the given circles as radius; draw *QV* parallel to *AB*, and at a distance from it equal to the radius of the smaller circle; find *S* the centre of the circle passing through C, D, and touching QV: this will be the required circle.

Because $SD = ST = SC$, and that $ED = IT = CG$, $\therefore$ (*Ax.* 3) $SE = SI = SG$; and since the lines SGC and $SEDK$ join the centres of the circles, they pass through the points of contact (III. 12); and since SI is perpendicular to AB, the circle GEL touches AB: hence GEL is the required circle.

EXERCISE 87.

Describe a circle which shall pass through a given point and touch a given circle at a given point.

(Fig. 87, Plate VI.)—Let P be the given point, BEF the given circle, and B the point in the circumference of the circle where the required circle shall touch BEF.

Join BP, bisect it, and from the point of bisection, D, draw DK perpendicular to BP, meeting CB produced at K: then K is the centre of the required circle.

BD and DK and the contained angle BDK are respectively equal to PD and DK and the contained angle PDK, $\therefore BK = PK$ (I. 4); and the circle described from K as centre, with KB as radius, will evidently be the required circle (III. 12).

EXERCISE 88.

To describe a circle which shall touch a given circle, and also touch a given straight line in a given point.

(Fig. 88, Plate VI.)—Let PFH be the given circle, and AB the given straight line, and D a given point in it; it is required to describe a circle which shall touch the circle PFH and also the straight line AB in the point D.

From D draw DE perpendicular to AB and equal to the radius of PFH; join C, the centre of PFH, and the point E; at C make an angle equal to the angle at E, and produce the lines forming these angles to meet in K; then K is the centre of the required circle.

Because the angles at C and E are equal, the line $CK = EK$ (I. 6); and as (*Cons.*) $CP = ED$, $\therefore KD = KP$. Again, since KD is perpendicular to AB, and that CK passes through the two centres, PWD is (III. 12, 19) the circle required.

PLATE V.

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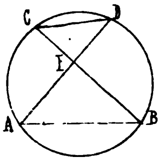

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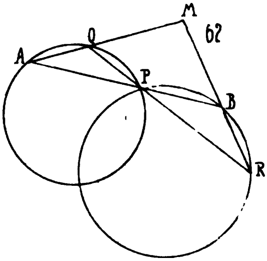

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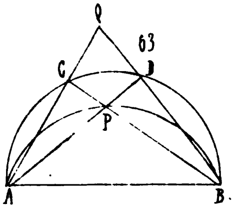

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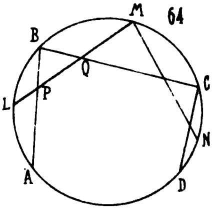

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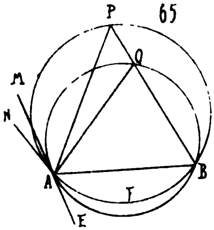

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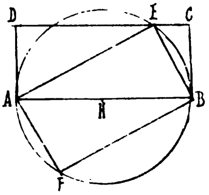

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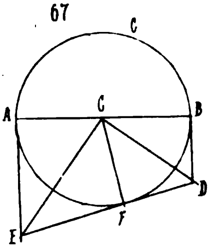

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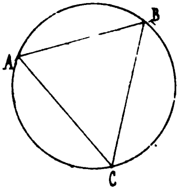

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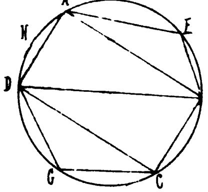

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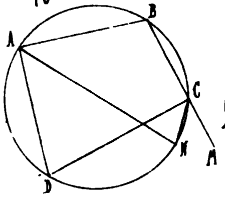

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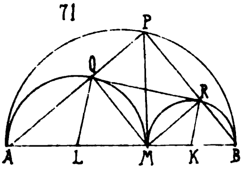

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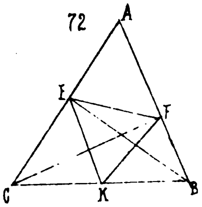

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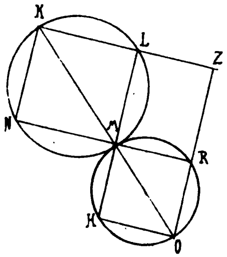

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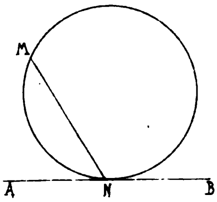

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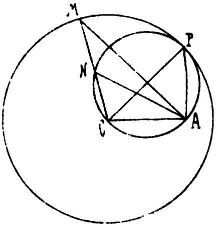

PLATE VI.

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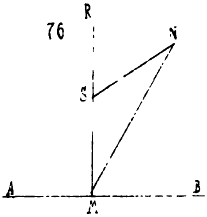

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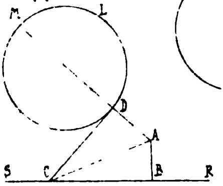

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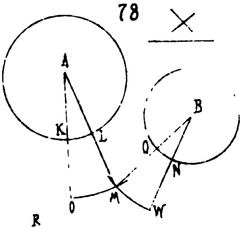

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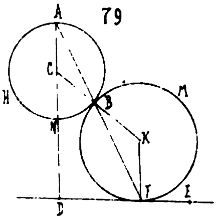

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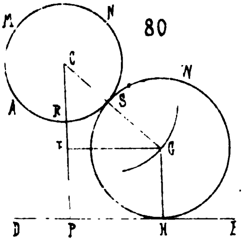

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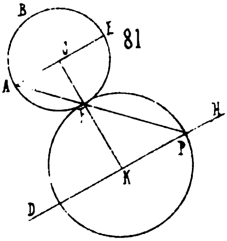

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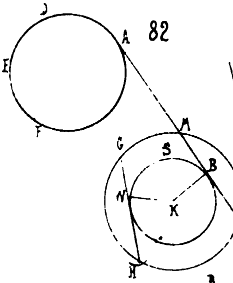

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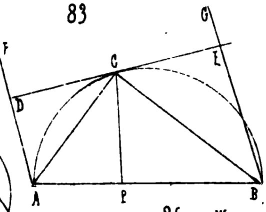

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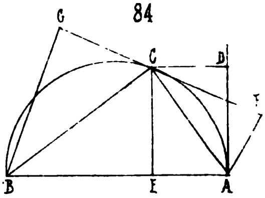

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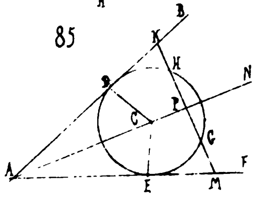

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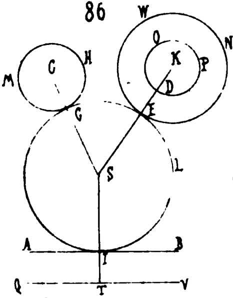

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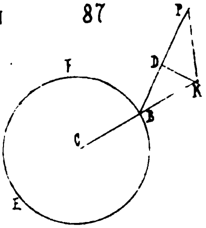

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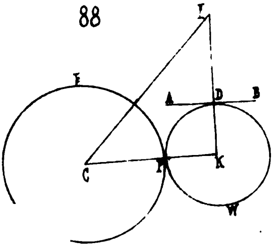

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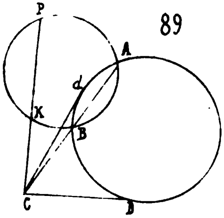

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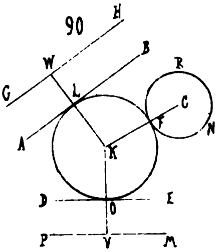

EXERCISE 89.

Through two given points to describe a circle which shall touch a given circle.

(Fig. 89, Plate VI.)—Let P and K be the given points. Take any point A on the circumference of the given circle, and describe a circle through P, K, and A. If this described circle touches the given circle, it is the required circle. But if not, let B be the other point of intersection of the two circles. Let PK and AB be produced to meet at C; and from C draw CD touching the given circle at D: then a circle described through P, K, and D will be the required circle.

Now, the rectangle $PC \cdot CK$ = rectangle $AC \cdot CB$ (III. 35); and the rectangle $AC \cdot CB = CD^2$ (III. 36); therefore the rectangle $PC \cdot CK = CD^2$ (*Ax.* 1). Hence the circle described through P and K to touch CD at D will be the required circle. If the tangent Cd be drawn, there will be two solutions.

EXERCISE 90.

To describe a circle touching two given straight lines and a given circle.

(Fig. 90, Plate VI.)—Let AB and DE be the given straight lines, and FRN the given circle; it is required to describe a circle touching AB, DE, and the circle FRN.

Draw GH and PM parallel to AB and DE respectively and at distances from them equal to the radius CF, of the given circle FRN; find the centre of a circle passing through C and touching GH and PM (*Ex.* 85); let K be that centre; draw CFK, and from K draw KLW and KOV perpendicular to GH and PM; with K as centre, and KF as radius, describe the circle LFO: then this is the required circle.

Because KC, KW, and KV are all equal; and CF, LW, and OV are also equal; then KF, KL, and KO are equal (*Ax.* 3). Again, because KL and KO are perpendicular to AB and DE, and the line KFC joins C and K, it follows (III. 12, 16, 19) that LFO is the required circle.

EXERCISE 91.

Through two given points to describe a circle which shall touch a given straight line.

(Fig. 91, Plate VII.)—Let C and D be the two given points, and AB the given straight line; it is required to describe a circle passing through C and D and touching AB.

Join CD, and produce it to meet AB in E; from EA or EB, produced if necessary, cut off a part EP, such that the square on EP shall be equal to the rectangle $CE{\cdot}ED$ (II. 14); through C, D, and P describe a circle CDP; this is the circle required.

Because the rectangle $CE{\cdot}ED = EP^2$, $\therefore$ EP touches the circle passing through C and D and meeting the line AB in P (III. 37). It is evident that, if the point P be taken on the other side of E, there will be two solutions.

EXERCISE 92.

In a given straight line to find a point such that the sum of its distances from two given points may be equal to a given straight line.

(Fig. 92, Plate VII.)—Let AB be the given straight line, and E and F the two given points; it is required to find a point in AB such that the sum of its distances from E and F may be equal to a given straight line.

From E draw EM at right angles to AB, cutting it in H, and make HM equal to HE; from F as centre, with distance equal to the given straight line, describe the circle ANS; describe a circle (*Ex.* 89) passing through E and M and touching the circle ANS, and let it touch it in N; join FN, cutting AB in C: then C is the required point. It is evident (III. 1 and 11) that C is the centre of EMN.

$CN = CE$; to each add CF, and $NF = EC + CF$; and $\therefore$ the point C, in AB, is the required point.

EXERCISE 93.

In a given straight line to find a point such that the difference of its distances from two given points may be equal to a given straight line.

(Fig. 93, Plate VII.)—Let AB be the given straight line, and E and F the two given points; it is required to find a point in AB such that its distances from E and F may be equal to a given straight line.

From F as centre, with radius equal to the given straight line, describe the circle MWS; from E draw EK at right angles to AB, cutting it in H; make HK equal to HE; through E and K describe a circle touching the circle MWS

in M (*Ex.* 89); join FM, and produce it to meet AB in C; then C is the required point. It is evident from (III. 1 and 12) that C is the centre of EKM.

Join EC.

$CE = CM$; and $CF - CM = CF - CE$ (*Ax.* 3) $= FM =$ the given straight line.

EXERCISE 94.

AB is a diameter of a circle, and AF is any chord; C is any point in AB, and through C a straight line is drawn at right angles to AB, meeting AF, produced if necessary at G, and meeting the circumference at D: show that the rectangle FA·AG, the rectangle BA·AC, and the square on AD are all equal.

(Fig. 94, Plate VII.)—Join BG; and produce DC to meet the circumference in H.

Because the angles at C and F are right angles, the circle described on BG as diameter will pass through C and F (III. 31); and ∴ the rectangle $GA \cdot AF =$ rectangle $BA \cdot AC$ (III. 36).

Again, $AD^2 = AC^2 + DC^2 = AC^2 + DC \cdot CH = AC^2 + AC \cdot CB$ (III. 35) $= BA \cdot AC$ (II. 3); but as $BA \cdot AC = GA \cdot AF$, ∴ $AD^2 = GA \cdot AF$ (*Ax.* 1).

If any other line be drawn from A cutting DH in X, and meeting the circumference in W: $WA \cdot AX = AD^2$. Join WB. The angle at W is right, being in a semicircle, and ∴ equals the angle at C, hence $WXCB$ is inscribable in a circle; ∴ $WA \cdot AX = BA \cdot AC = AD^2$.

EXERCISE 95.

Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls.

(Fig. 95, Plate VII.)—On the given base AB describe the arc ACB containing an angle equal to the given angle (III. 33), and from D, the given point, draw DC perpendicular to AB cutting the arc in C; join AC, BC: ACB is evidently the triangle required.

EXERCISE 96.

Construct a triangle having given the base, vertical angle, and perpendicular.

(Fig. 96, Plate VII.)—On the base *AB* describe the segment *ACB* containing an angle equal to the given angle (III. 33); draw *BD* at right angles to *AB* and equal to the given perpendicular; draw *DC* parallel to *AB* meeting the arc at *C*, and join *AC*, *BC*: *ABC* is the triangle required (*Cons.*).

Note.—This is impossible when the given perpendicular is greater than that drawn from the middle of *AB* to meet the arc.

EXERCISE 97.

Construct a triangle, having given the base, vertical angle, and the median.

(Fig. 97, Plate VII.)—On the base *AC* describe the segment *ABC* containing an angle equal to the given angle (III. 33), and with the centre of the given base as centre, and the given median as radius, describe an arc cutting *ABC* in *B*; join *BA*, *BC* : *ABC* is the triangle required (*Cons.*).

Note.—A median of a triangle is the line drawn from an angle to the point of bisection of the opposite side.

EXERCISE 98.

Given base, altitude, and radius of circumscribed circle, construct the triangle.

(Fig. 98, Plate VII.)—With the given radius *SK* describe the circle *KLA*, and with *K* as centre, and the given base as distance, describe an arc cutting the circumference at *A*, and join *KA*; from *A* draw *AB* at right angles to *KA*, and equal to the given altitude; draw *BL* parallel to *KA* cutting the circumference in *L*, and join *LA*, *LK*: *KLA* is evidently the triangle required.

EXERCISE 99.

Given the base of a triangle, the vertical angle, and the radius of the inscribed circle: to construct the triangle.

(Fig. 99, Plate VII.)—On the given base *AB* describe a segment of a circle containing an angle equal to the sum of the vertical angle and half its supplement; at a distance from *B* equal to the given radius draw a straight line parallel to *AB*, and meeting the arc in *C*; from *C* let fall the perpendicular *CD*; join *BC* and *AC*; make the angle *CAF* equal to the angle *BAC*, and the angle *CBE* equal to the angle *CBA*; produce *BE* to meet *AF*: then *AFB* is the triangle required.

Because the angle *ACB* is equal to the vertical angle and half its supplement, the two angles BAC and ABC = half the supplement (I. 32); and ∴ their equals, the angles FAE and CBF are together equal to half the supplement. But $ACB = CEB + CBE = CBE + FAE + AFB$ (I. 32), because $CEB = FAE + AFB$; and ∴ $AFB + FAE + FBC$ = to the vertical angle and half its supplement. From these equals take FAE and FBC, which are equal to half the supplement, and the angle AFB = the vertical angle.

Again, draw CH, CN perpendicular to AF, FB; and since the angle HAC = angle DAC, the angles at D and H right angles, and AC common to both triangles, the line $CD = CH$; and in like manner it may be shown that $CN = CD$; and ∴ $CN = CH$. C is therefore the centre of the inscribed circle (III. 9), and ABF is the required triangle.

EXERCISE 100.

Given the vertical angle of a triangle, and the segments into which the line bisecting the vertical angle divides the base: to construct the triangle.

(Fig. 100, Plate VII).—Let AE and EB be the segments of the base made by the line bisecting the vertical angle; on AB, their sum, describe a segment of a circle containing an angle equal to the given vertical angle (III. 33), and complete the circle; bisect AB in C, and draw CD at right angles to AB meeting the arc in D; through D and E draw the straight line DEF; join AF and BF; then AFB is the triangle required. The proof is evident from the construction.

EXERCISE 101.

Given the three lines drawn from the vertex of a triangle, one of them perpendicular to the base, one bisecting the base, and one bisecting the vertical angle: to construct the triangle.

(Fig. 101, Plate VII.)—Take any straight line AB, and from any point, E, draw EF at right angles to AB and equal to the given perpendicular; from F as centre, with distance equal the line bisecting the vertical angle, describe an arc cutting AB in H: from the same point as centre, with distance equal to the line bisecting the base, describe an arc cutting AB in M; through M draw the straight line NMG at right angles to AB; at F, with the line FH, make the angle HFK equal to the angle FGK, contained by FH and KM produced;

from *K* as centre, with *KF* or *KG* as radius, describe a circle cutting *AB* in *C* and *D*; join *FC* and *FD*: then *CFD* is the triangle required.

Because *NKM* is perpendicular to *CD*, it also bisects it (III. 3), and the arcs *CG* and *GD* are equal (III. 30); and ∴ the angle *CFG* = angle *GFD* (III. 27). Now, in the triangle *CFD*, the perpendicular *FE*, the line *FH* bisecting the vertical angle, and the line *FM* bisecting the base, are equal to the given lines; and ∴ *CFD* is the required triangle.

EXERCISE 102.

Given the vertical angle, the difference of the sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex: to construct the triangle.

(Fig. 102, Plate VII.)—Let *AD* equal the difference of the segments of the base, the angle *ADF* equal to half the vertical angle, and *AF* equal to the difference of the sides. Produce *AF*, and make the angle *CDF* equal to the angle *CFD*; produce *AD*; from *C* as centre with *CD* or *CF* as radius, describe the circle *FDBE*, cutting *AD* and *AC* produced in *B* and *E*; join *EB* and *BC*: then *ABC* is the triangle required.

Because *BDFE* is a quadrilateral inscribed in a circle, the angles *BEF* and *FDB* are equal to two right angles (III. 22); but the angles *ADF* and *FDB* are also equal to two right angles (I. 13), ∴ the angle *ADF* = angle *BEF* (*Ax.* 3), and as the angle *ACB* is double of the angle *BEF*, it is also double of *ADF*. It is therefore evident (*Cons.*) that *ABC* is the triangle required.

EXERCISE 103.

Given the sum of the legs of a right-angled triangle, and the sum of the hypotenuse and the perpendicular to it from the right angle: to construct the triangle.

(Fig. 103, Plate VII.)—Let *FG* be equal to the sum of the legs, and *CB* equal to the sum of the hypotenuse and perpendicular; it is required to construct the triangle.

Find a straight line whose square is equal to $CB^2 - FG^2$ (I. 47); from *CB* cut off *CA* equal to that line; on *AB* describe a semicircle; from *B* raise *BH* perpendicular to *AB* and equal to AC; from H draw HE parallel to AB; and from either point of intersection—namely, from E or M, draw a perpendicular to AB; join AE and EB, or AM and MB; then AEB or AMB is the triangle required.

$CB^2 = FG^2 + AC^2 = FG^2 + ED^2$ (*Cons.*); but $CB^2 = (CA + AB)^2 = (ED + AB)^2 = ED^2 + AB^2 + 2ED \cdot AB$ (II. 4); and $\therefore FG^2 + AC^2 = FG^2 + ED^2 = ED^2 + AB^2 + 2ED \cdot AB$ (*Ax.* 1); hence $FG^2 = AB^2 + 2ED \cdot AB$ (*Ax.* 3). Again, $FG^2 = (AE + EB)^2$ (*Hyp.*) $= AE^2 + EB^2 + 2AE \cdot EB$ (II. 4); and $\therefore AE^2 + EB^2 + 2AE \cdot EB = AB^2 + 2ED \cdot AB$; but $AE^2 + EB^2 = AB^2$ (III. 31 and I. 47), $\therefore 2AE \cdot EB = 2ED \cdot AB$ (*Ax.* 3), that is, $AE \cdot EB = ED \cdot AB =$ twice area of triangle AEB; hence $(AE + EB)^2 = FG^2$; and $\therefore AE + EB = FG$, and AEB is the triangle required.

EXERCISE 104.

Construct a triangle, having given the base, the vertical angle, and the length of the line drawn from the vertex to the base bisecting the vertical angle.

(Fig. 104, Plate VII.)—On the given base AB describe the segment ACB containing an angle equal to the given angle (III. 33), and complete the circle.

Bisect the arc AKB in K, and join KB; produce RS, the given line, to H, making $RH \cdot HS = KB^2$, and with K as centre and radius equal to HS, describe an arc cutting AB in L; join KL and produce it to meet the circumference at C; join CA, CB: ABC is the triangle required.

$CK \cdot KL = KB^2$ (*Ex.* 94), $\therefore RH \cdot HS = CK \cdot KL$, but $HS = KL$ (*Cons.*), $\therefore CL = RS$ or the given bisector; and since CK bisects the arc AKB, it also bisects the angle at C.

EXERCISE 105.

Given the perimeter, altitude, and vertical angle: to construct the triangle.

(Fig. 105, Plate VII.)—Let BG, BF, the halves of the given perimeter, be placed so as to contain an angle equal to the given angle, and let GEF be the circle to which these are tangents; with B as centre, and the given altitude as distance, describe the circle RDL; draw the common tangent $AEDC$, touching the circles in E and D, and the tangents in A and C: ABC is the triangle required.

Join BD.

BD is at right angles to *AC*, and equals the given altitude. Again, *AE* = *AG*, and *CE* = *CF* being tangents, ∴ the perimeter of *ABC* equals the given perimeter; hence *ABC* is the triangle required.

EXERCISE 106.

Given the perimeter, the vertical angle, and the line bisecting the vertical angle : construct the triangle.

(Fig. 106, Plate VIII.)—Let *BA*, *BC*, the halves of the given perimeter, be placed so as to contain an angle equal to the given angle, and let *AFC* be the circle to which they are tangents; draw *BM* bisecting the vertical angle and equal to the given bisector; draw the tangent *DFME*, meeting the tangents at *D* and *E*: *DBE* is the triangle required.

Because *BM*, the given bisector, bisects the given angle *DBC*, and *DF* = *DA*, and *EF* = *EC*, the perimeter of triangle *DBE* = the given perimeter, and *DBE* is the triangle required.

EXERCISE 107.

Given the base and vertical angle of a triangle: to find the locus of the vertex.

(Fig. 107, Plate VIII.)—On the given base *AB* describe the segment *ACB* containing an angle equal to the given angle (III. 33): the arc *ACB* is the required locus, for any triangle *ACB* having its vertex in the arc satisfies the required conditions.

EXERCISE 108.

Given the base and vertical angle of a triangle: to find the locus of the intersection of the perpendiculars on the sides from the opposite angles.

(Fig. 108, Plate VIII.)—Let *ABC* be a triangle satisfying the necessary conditions, *AC* being the given base, and *ABC* the given angle; *AE*, *BD*, and *CF* perpendiculars on the sides *BC*, *AC*, and *AB* respectively, intersecting in the point *M*.

Now, since the angles at *F*, *E* are right angles, the angles *FBE*, *FME* are also equal to two right angles, but *FBE* is given, and ∴ *FME* or *AMC* is constant, hence the locus of the intersection of the perpendiculars is in segment described on *AC*, the given line, containing an angle equal to the supplement of the given angle.

EXERCISE 109.

Given the base and vertical angle of a triangle: to find the locus of the intersection of the bisectors of the angles.

(Fig. 109, Plate VIII.)—Let ABC be a triangle whose base AC equals the given base, and angle ABC the given angle; let the bisectors of the angles at A and C meet at AD.

The angle ADC is constant because it is equal to half the sum of the angles at A and C, together with the angle at B (**I. 32**), and ∴ the required locus is in the arc of the segment described on AC containing an angle equal to the given angle and half its supplement.

EXERCISE 110.

AB is a fixed chord of a circle, AC is a movable chord of the same circle; a parallelogram is described of which AB and AC are adjacent sides: find the locus of the middle points of the diagonals of the parallelogram.

(Fig. 110, Plate VIII.)—Let AB and AC be the two chords in the circle ACB, AB being fixed and AC movable; and let $ACNB$ be a parallelogram; it is required to find the locus of the middle point of the diagonals.

Join B and K the centre of the circle, and on BK as diameter describe a circle meeting the line AN in M; the point M is the middle point of both diagonals and its locus is in the circumference of KMB.

Because the circle KMB is described on the radius of the circle ACB, the chord BC from B is bisected (*Ex.* 4) in M; that is, $BM = CM$. But the diagonals of every parallelogram bisect each other (**I. 34**); and therefore AN is also bisected in M.

Note.—The diagonal AN will be a maximum when it passes through the centre of the smaller circle; for it will be double of AW, which (**III. 8**) is the longest line from A to any point in the circle $BWMK$.

EXERCISE 111.

Let O and C be any fixed points on the circumference of a circle, and OA any chord: then if AC be joined and produced to B, so that OB is equal to OA, the locus of B is an equal circle.

(Fig. 111, Plate VIII.)—Join OC.

Because the angle at A = angle at B, whatever segment B

is in on the line *OC*, is similar to the segment *OAC* (*Def.* 11); and similar segments on equal straight lines, or on the same straight line, are equal (III. 24).

EXERCISE 112.

Find a point in a given straight line from which the tangent drawn to a given circle is of a given length.

(Fig. 112, Plate VIII.)—Let *AB* be a given straight line, and *HDE* a given circle; it is required to find a point in *AB* from which the tangent to *HDE* shall be equal to a given straight line.

From *C*, the centre of the given circle, with distance equal to the line, the square on which is equal to sum of the squares on the radius of *HDE* and the given line, describe a circle cutting the line *AB* in the point *P*: then *P* is the required point.

Join *CP*; from *P* draw the tangent *PD*, and join *CD*.

$CP^2 = CD^2 + DP^2$ (I. 47), and since *CD* is a radius, *PD* equals the given line.

EXERCISE 113.

If a circle cuts another circle given in position at right angles, and also passes through a fixed point, prove that it must pass through another fixed point.

(Fig. 113, Plate VIII.)—Let *ABPM* be a circle cutting another circle *AHB*, given in position, at right angles, and pass through the fixed point *P*; it must also pass through another fixed point *E*.

AC is (*Hyp.*) a tangent to the circle *ABPM*, and ∴ the rectangle $PC \cdot CE = AC^2$ (III. 36); and as the circle *ABPM* must pass through the fixed point *P* (*Hyp.*), and *P* and *C* are joined, the point *E* is also a fixed point; for the line *PC* cannot cut the circle in any other point.

EXERCISE 114.

From a given point without a circle, at a distance from the circumference not greater than its diameter, draw a straight line to the concave circumference which shall be bisected by the convex circumference.

(Fig. 114, Plate VIII.)—Let P be the given point, and HQM the given circle.

On the tangent PQ describe the right-angled isosceles triangle PXQ; produce PX to L making $XL = XQ$, and with P as centre, and PL as distance, describe an arc cutting the circumference in H; draw PH cutting the circumference in M: PH is the line required.

$PQ^2 = PX^2 + QX^2 = LP \cdot PX = HP \cdot PM$ (I. 47), and (III. 36), but $PL = PH$, $\therefore$ $PX = PM$, hence PH is bisected in the point M.

EXERCISE 115.

E is the middle point of a semicircular arc AEB, and CDE is any chord cutting the diameter at D, and the circle at C: show that the square on CE is twice the quadrilateral $AEBC$.

(Fig. 115, Plate VIII.)—From C draw CH parallel to AB, and meeting EK produced at H; and join CK.

$CE^2 = CH^2 + EH^2$ (I. 47); but $EH^2 = EK^2 + KH^2 + 2EK \cdot KH$ (II. 4); therefore $CE^2 = CH^2 + EK^2 + KH^2 + 2EK \cdot KH$ (*Ax.* 1). Again, $CH^2 + KH^2 = CK^2$ (I. 47), $\therefore$ $CE^2 = EK^2 + CK^2 + 2EK \cdot KH = 2AK^2 + 2EK \cdot KH$.

Now, $2AK^2 = 2AEB$ (I. 41); and $2EK \cdot KH = AB \cdot KH = 2ACB$; and $\therefore$ $CE^2 = 2AEB + 2ACB$ = twice the quadrilateral $AEBC$ (*Ax.* 1).

EXERCISE 116.

The circle described through any two of the angular points of a triangle and the intersection of the perpendiculars from the angles on the opposite sides will be equal to the circumscribing circle of the triangle.

(Fig. 116, Plate VIII.)—Let ABC be a triangle, and let AF, BD, CE be perpendiculars on the sides from the opposite angles, intersecting in G; let the circle $BGCH$ be described through the points B, G, C; then the circle $BGCH$ will be equal to the circle circumscribing the triangle ABC.

Because the angles at D and E are right angles, the sum of the angles EAD and EGD = two right angles; but angle EGD = angle BGC (I. 15), $\therefore$ angles BAC and BGC = two right angles (*Ax.* 1). Again, the angles BHC and BGC are = two right angles (III. 22), $\therefore$ the angles BAC and BGC = the angles *BHC* and *BGC* (*Ax.* 1), and hence the angle $BAC = BHC$, and the segment *BAC* is similar to the segment *BHC*, and, being on the same straight line *BC*, they are equal **(III. 24)**; and the entire circle *BAC* = circle *BHC*.

EXERCISE 117.

If AB and CD be two chords of a circle which cut at right angles at E, prove that the sum of the arcs AD and BC is equal to the sum of the arcs AC and BD.

(Fig. 117, Plate VIII.)—Because the angle *AED* is measured by half the sum of the two arcs *AD* and *BC*, and the angle *BED* is measured by half the sum of the two arcs *AC* and *BD*, the sum of the two arcs *AD* and *BC* = sum of the two arcs *AC* and *BD*, the angles at *E* being right angles.

EXERCISE 118.

Through B, a point inside a circle DCN, draw a line so that the difference of the segments of the intercepted chord shall be equal to a given line.

(Fig. 118, Plate VIII.)—Join *B* with the centre *A*, and with *AB* as radius, describe the circle *MBR*; with *B* as centre, and the given line as radius, describe an arc cutting the circumference of *MBR* in *M*; join *BM*, and produce it both ways to meet the outer circumference at *D* and *C*: *DBC* is the line required.

For, draw *AL* perpendicular to *DC*; then **(III. 3)** $LD = LC$, and $LM = LB$, $\therefore DM = BC$; hence the difference between *DB* and $BC = MB$, or the given line.

EXERCISE 119.

If a quadrilateral be described about a circle, the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

(Fig. 119, Plate VIII.)—Let *ABDE* be a quadrilateral described about the circle *FGHK*, and let the angular points be joined with the centre *C*; then the angles *ACB* and *ECD* are together equal to two right angles.

Because the angles of the quadrilateral, which are equal to four right angles, are bisected by the lines *AC*, *BC*, *DC*, and

PLATE VII

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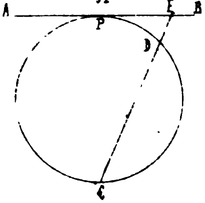

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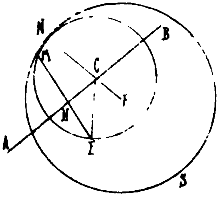

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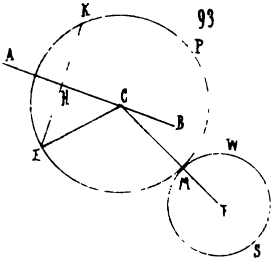

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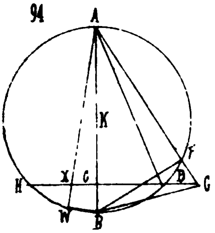

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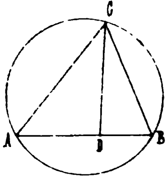

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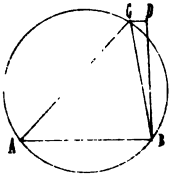

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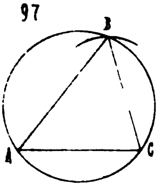

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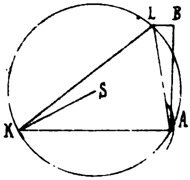

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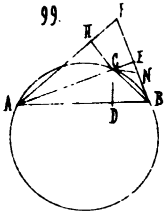

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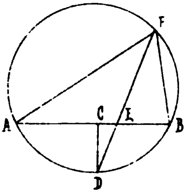

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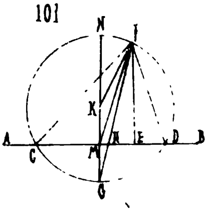

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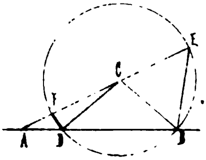

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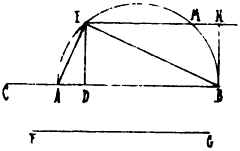

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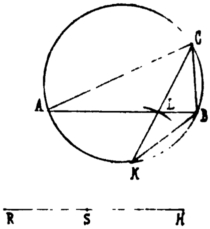

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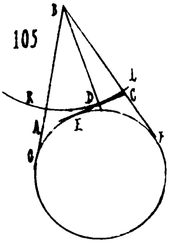

PLATE VIII

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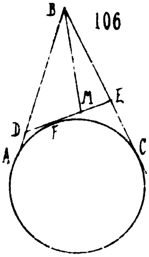

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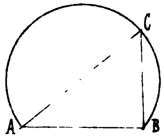

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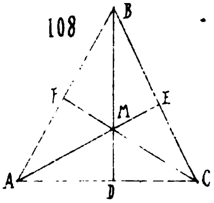

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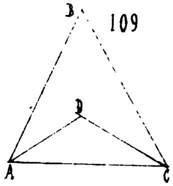

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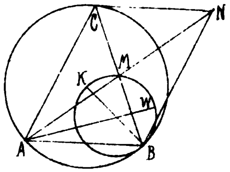

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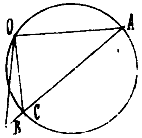

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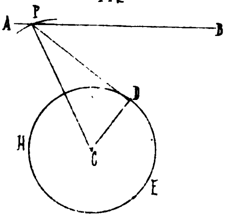

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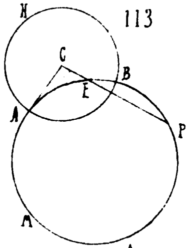

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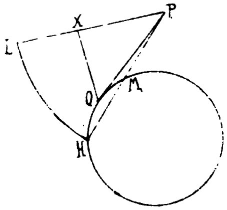

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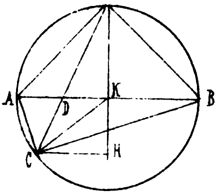

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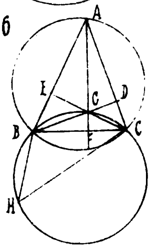

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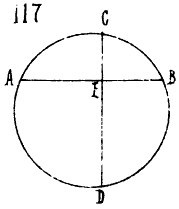

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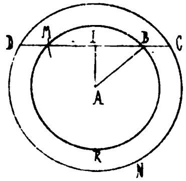

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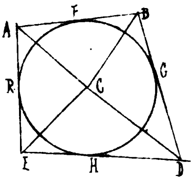

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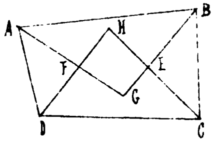


EC, their halves are together equal to two right angles, ∴ the angles *BAC*, *ABC*, *CED*, and *CDE* are together equal to two right angles; but the angles *BAC*, *ABC*, and *ACB* are also equal to two right angles (I. 32), ∴ the angle *ACB* = the two angles *CED* and *CDE*. In the same manner it will at once be seen that the angle *ECD* = the two angles *CAB* and *CBA*, ∴ the two angles *ACB* and *ECD* are together equal to the angles *ABC*, *BAC*, *EDC*, and *DEC*, and hence *ACB* and *ECD* are together equal to two right angles.

EXERCISE 120.

Show that the four straight lines bisecting the angles of any quadrilateral form a quadrilateral which can be inscribed in a circle.

(Fig. 120, Plate VIII.)—Let *ABCD* be a quadrilateral, its angles being bisected by *AG*, *BG*, *CH*, *DH*; and let *AG*, *DH* meet in *F*, and *BG*, *CH* in *E*. In the triangle *AFD*, the three angles are (I. 32) equal to two right angles; and in the triangle *BEC*, the three angles are, in like manner, equal to two right angles; but (*Hyp.*) the four angles *FAD*, *FDA*, *EBC*, and *FCB* are together equal to two right angles, being equal to half the four angles of *ABCD*; hence the two angles at *E* and *F* are equal to two right angles, and ∴ the two angles *EHF* and *EGF* are together equal to two right angles, and a circle can be described about the figure *EGFH* (III. 22).

EXERCISE 121.

If two circles can be described so that each touches the other and three of the sides of a quadrilateral figure, then the difference between the sums of the opposite sides is double the common tangent drawn across the quadrilateral.

(Fig. 121, Plate IX.)—Let *ABCD* be a quadrilateral figure, and let the two circles *HKMS* and *GSNP* touch each other, and let each also touch three sides of the quadrilateral; then $(AB + DC) — (AD + BC) = 2EF$, the common tangent at the point of contact drawn across the quadrilateral.

Because *AEFD* is a quadrilateral described about a circle, $AE + DF = AD + EF$; for the same reason $EB + FC = EF + BC$; and ∴ (*Ax.* 1) $AE + DF + EB + FC = AD + BC + 2EF$, or $AB + DC = AD + BC + 2EF$; and ∴ $2EF$ is equal to the difference between the sum of *AB* and *DC* and the sum of *AD* and *BC*.

EXERCISE 122.

A circle is described on the shorter diagonal of a rhombus as a diameter, and cuts the sides; and the points of intersection are joined crosswise with the extremities of that diagonal: show that the parallelogram thus formed is a rhombus with angles equal to those of the first.

(Fig. 122, Plate IX.)—Let *ABCD* be the rhombus whose shorter diagonal is *AC* upon which as diameter the circle *AEFCGH* is described. Because *AD* = *CD*, the angle *DAC* = angle *DCA* (I. 5); and the angle *HAG* = angle *HCG* (III. 21), ∴ the angle *MAC* = angle *MCA* (*Ax.* 3), and the side *MA* = side *MC* (I. 6). In like manner it will be seen that *KA* = *KC*. Again, because the angle at *B* = angle at *D* (I. 34), the angles *BFA* and *DGA* right angles (III. 31), and the side *BA* = side *DA* (*Hyp.*), the triangles *DAG* and *BAF* are equal in every respect (I. 26), and the side *BF* = side *DG*; but (*Hyp.*) *DC* = *BC*, ∴ *GC* = *FC* (*Ax.* 3), and the arc *GNC* = arc *CWF* (III. 28), and the angle *GAC* = angle *FAC* (III. 27), ∴ the triangles *AMC* and *KAC* are equal in every respect (I. 26), and the sides *AK* and *KC* are equal to *MA* and *MC*, the angle *MAK* = angle *MCK*, and the angle *AMC* = angle *AKC* (I. 34). Again, the angles *HDG* and *HMG* or *AMC* are together equal to two right angles, and the angles *KAM* and *AMC* are also equal to two right angles (I. 29), ∴ *KAM* = *ADC* (*Ax.* 1), and *DAB* = *AMC*. Therefore *AMCK* is a rhombus whose angles are equal to those of *ABCD*.

EXERCISE 123.

If two circles cut each other, any two parallel straight lines drawn through the points of sections to cut the circles are equal.

(Fig. 123, Plate IX.)—Let *AEFC* and *EFDB* be two circles cutting each other in the points *E* and *F*; and let *AB* and *CD* be parallel chords passing through *E* and *F*: then *AB* and *CD* are equal.

Draw *BD*, *EF*, *AC*.

Because *AEFC* is a quadrilateral inscribed in a circle, *EFD* = *EAC*; for a similar reason *EFC* = *EBD*, ∴ *EAC*, *EBD* equal two right angles, and ∴ *AC* is parallel to *BD* (I. 28), hence *ABDC* is a parallelogram, and *AB* = *CD* (I. 34).

Otherwise—From the centres *M*, *N* of the circles let fall the perpendiculars *MK*, *MQ*, *NR*, *NH* on the parallels, and let the points *K*, *R* be on the line *AB*.

MQ, MK are in the same straight line, because they are perpendiculars to the parallels; for a similar reason FXR is a straight line, $\therefore$ $QKRH$ is a rectangle, and $KR = QH$ (I. 34), but since the perpendiculars bisect the chords on which they fall (III 3), $\therefore$ AB is double of KR, and CD is double of QH, $\therefore AB = CD$.

EXERCISE 124.

Two circles whose centres are A and B intersect at C; through C two chords DCE and FCG are drawn equally inclined to AB, and terminated by the circles: show that DE and FG are equal.

(Fig. 124, Plate IX.)—From A and B draw AH, BN perpendicular to DE and AM, BK perpendicular to FG; from A, B draw AO, BP parallel to FG and DE respectively.

Because $DE = 2HN$ (III. 3), and $FG = 2MK$, and that $HN = BP$, and $MK = AO$; $\therefore DE = 2BP$, and $FG = 2AO$. Again, since DE and FG are equally inclined (*Hyp.*) to AB, their parallels BP and AO are equally inclined to AB, $\therefore$ the angle PBA = angle OAB. But the angles APB and AOB are (*Cons.*) right angles, and the side AB is common to the two triangles ABP and OAB, $\therefore$ (I. 26) $BP = AO$; and since $DE = 2BP$, and $FG = 2AO$, $\therefore$ (*Ax.* 6) $DE = FG$.

EXERCISE 125.

Through either of the points of intersection of two given circles to draw the greatest possible straight line terminated both ways by the two circumferences.

(Fig. 125, Plate IX.)—Let $ECWN$ and $DFSN$ be two circles intersecting each other in N and P; it is required to draw either through N or P the longest possible straight line terminated both ways by the circumferences of both circles.

Join A and B, the centres of both circles; through P, one of the points of intersection, draw CD parallel to AB; then CD is the longest possible straight line that can be drawn through P and terminated both ways by the circumferences. Let EF be any other such line; CD is greater than EF. From A, B draw AG, BM perpendicular to CD, and AK, BH perpendicular to EF; and from B draw BT parallel to EF, meeting AK in T. The angle BTA is a right angle, and (I. 17) the angle BAT is less than a right angle, $\therefore$ AB is greater than *BT* (I. 19); but *AB* = *GM*, and *TB* = *KH* (I. 34), and ∴ *GM* is greater than *KH*; and since *CD* = 2*GM* (III. 3), and *EF* = 2*KH*, ∴ *CD* is greater than *EF*; hence *CD* is the greatest possible straight line that can be drawn through *P*, and, etc.

EXERCISE 126.

Draw through one of the points in which any two circles cut one another a straight line which shall be terminated by their circumferences and bisected in their point of section.

(Fig. 126, Plate IX.)—Bisect the line *AB*, joining their centres, in the point *S*; join *S* with one of their points of intersection *L*, and through *L* draw *NLR* at right angles to *SL* terminated by the circles at *N* and *R*: *NR* is the line required.

Let fall the perpendiculars *AC*, *BD*, on *NR*, and draw *BM* at right angles to *AC*.

Because *SL* is drawn parallel to *AC* from the point of bisection of *AB*, it ∴ bisects *MB* or *CD* (*Ex.* 133, Part I.), but *NL* is double of *CL* and *RL* double of *DL* (III. 3), ∴ *NL* = *RL*.

EXERCISE 127.

Two circles cut each other at a point A: it is required to draw through A a straight line so that the extreme length of it intercepted by the two circles may be eqnal to that of a given straight line.

(Fig. 127, Plate IX.)—Join *C* and *K* the centres of the two circles; and on *CK* describe a semicircle. From *K*, with distance equal to half the given line, describe an arc cutting the arc of the semicircle in *D*; through *A* draw *EF* parallel to *DK*: then *EF* is equal to the given straight line. From *K* draw *KH* perpendicular to *FE*; join *CD*, and produce it to meet *FE* in *G*.

The angle *CDK*, in a semicircle, is a right angle (III. 31); and since *EF* is parallel to *DK* the angle *CGE* is also a right angle (I. 29); and since (III. 3) *CG* and *KH* bisect *FA* and *AE*, *FA* + *AE*, or *FE* is double of *KD*; and ∴ *FE* = the given straight line.

EXERCISE 128.

Draw a right line through a point of intersection of two circles, so that the difference of the intercepted segments shall be equal to a given straight line.

(Fig. 128, Plate IX.)—With *D*, one of the points of intersection of the given circles *NDQ*, *SDQ*, whose centres are *A*, *B*, respectively, as centre, and half the given difference as radius, describe a circle; and from *M*, the middle point of *AB*, draw a tangent touching this in *C*; join *CD*, and produce it both ways meeting the circles *NDQ*, *SDQ* in *N*, *S* respectively: *NS* is the line required.

Draw *AR*, *BK* perpendicular to *NS*.

Because *MC* is a tangent, and is drawn from the middle point of *AB*, ∴ *MCD* is a right angle, and $RC = CK$, hence *CD* is the difference between *RD* and *DK*, ∴ the difference between *ND* and *DS* equals twice *CD*, or the given line.

EXERCISE 129.

Draw a right line through a point of intersection of two circles, so that the rectangle under the intercepted segments shall be equal to a given quantity.

(Fig. 129, Plate IX.)—Let *L*, *K* be the points of intersection of the circles *LKA*, *LKB*, and let *K* be the point through which the line is to be drawn.

Join *LK* and produce it to *M*, making $LK \cdot KM$ equal to the given quantity; draw *MA* meeting *LKA* in *A*, and making the angle at *M* equal to the angle on the segment *LBK*; join *AK* and produce it to meet *LKB* in *B*: *AKB* is the line required.

Join *LB*.

The angle at *M* equals the angle at *B* (*Cons.*), ∴ the points *A*, *L*, *B*, *M* are in the circumference of a circle, and hence (III. 35) $AK \cdot KB$ equals $LK \cdot KM$ or the given quantity; ∴ *AKB* is the line required.

EXERCISE 130.

Draw from a given point in the circumference of a given circle, a chord which shall be bisected by its point of intersection with a given chord of the circle.

(Fig. 130, Plate IX.)—Let *PAFB* be the given circle, *AB* a given chord, and *P* a given point in the circumference; it is required to draw from *P* a chord which shall be bisected by the chord *AB*.

Join P and C the centre of the given circle; on PC describe a semicircle, cutting the chord AB in E; through E draw PEF; then PF is bisected in E. Draw CE.

Because $PGEC$ is a semicircle, the angle PEC is a right angle (III. 31); and $\therefore$ the chord PF is bisected at E by EC (III. 3).

As the arc of the semicircle cuts the chord AB in two places, as at E and G, there can then be drawn two chords from the given point P, each of which shall be bisected in these two points E and G. If the arc of the semicircle only touch the chord AB, there can be only one solution.

EXERCISE 131.

To draw a straight line cutting two concentric circles, so that the part of it intercepted by the circumference of the greater may be double the part intercepted by that of the less.

(Fig. 131, Plate IX.)—Let FDN and EAM be two concentric circles, C being the centre; it is required to draw a straight line cutting both circles, so that the chord of the larger circle may be double of the chord of the smaller circle.

From C draw any radius CA, which produce, making the part produced, AB, equal to CA; on AB describe the circle ADB, cutting, when possible, the circle FDN in D; draw the straight line $DAEF$, which will be the required straight line. From C let fall CH perpendicular to AE.

In the triangles ADB and ACH, the angles at A are equal (I. 15), the angles at D and H are right angles, and $AC = AB$, $\therefore AD = AH$ (I. 26) $= HE$; but $AD = EF$, hence $AD + EF = AE$; and $\therefore DF = 2AE$.

EXERCISE 132.

In the circumference of a given circle determine a point so situated that if chords be drawn from the extremities of a given chord of the circle their difference shall be equal to a given straight line less than the given chord.

(Fig. 132, Plate IX.)—Let AC be the given chord, and ABC the given circle.

On AC describe the segment AKC containing an angle equal to the angle in the segment ABC, together with half its supplement; with A as centre, and the given difference as radius, describe an arc cutting AKC in K; join AK and produce it to meet the outer circumference at B: B is the point required.

$AKC = ABC + BCK$ (I. 32), $\therefore$ BCK equals half the complement of ABC, and $\therefore$ equals BKC which must be the remaining half complement; hence $BK = BC$ (I. 6), $\therefore$ the difference between BA, BC equals AK, or the given difference.

EXERCISE 133.

One circle touches another internally at the point A: it is required to draw from A a straight line such that the part of it between the circles may be equal to a given straight line, which is not greater than the difference between the diameters of the circles.

(**Fig. 133, Plate IX.**)—Let AB be the diameter of the outer circle and AE that of the inner, these being in the same straight line (III. 6). On BE describe a semicircle BKE, and with E as centre, and the given line as distance, describe an arc cutting the semicircle in K; join EK, BK and produce BK to meet the outer circumference at C, and draw CA cutting the circumference of the less circle at D: CD is the line required.

For since the angles at K, C, D are right angles (III. 31), $KCDE$ is a rectangle, and $\therefore$ $CD = KE$ (I. 34) $=$ the given line.

EXERCISE 134.

If from the extremities of the diameter of a semicircle perpendiculars be let fall on any line cutting the semicircle; the parts intercepted between those perpendiculars and the circumference are equal, and the smaller perpendicular is equal to that part of the greater which is exterior to the circumference.

(**Fig. 134, Plate IX.**)—Let AD, BKR be the perpendiculars on DR, which cuts the arc in N and Q: $DN = QR$ and $AD = KR$.

Join AK, and draw CML perpendicular to DR cutting AK in M.

The angles at K are right (III. 31), $\therefore$ AR is a rectangle, $\therefore$ $AD = KR$; and since CL bisects NQ (III. 3) it also bisects AK or DR, $\therefore$ $DN = QR$.

EXERCISE 135.

If perpendiculars AR, BS *be let fall from the extremities of a diameter* AB *of the circle* ADB, *on the chord* CD *which cuts the diameter: the intercepts* CR *and* DS *are equal.*

(Fig. 135, Plate IX.)—Draw SA, and through K, the centre, draw MKL parallel to AR or BS, meeting SA and CD in M and L respectively.

In the triangle ABS, because KM is drawn from the middle point of AB parallel to the base BS, it bisects the side SA; and for a similar reason, in triangle ASR, ML drawn from the middle point of SA parallel to the base AR, bisects SR; but CD is also bisected in the point L, KL being at right angles to it from the centre, $\therefore$ the parts CR, DS are equal.

EXERCISE 136.

Through a point C *in the circumference of a circle two straight lines* ACB *and* DCE *are drawn cutting the circle at* B *and* E: *show that the straight line which bisects the angles* ACE *and* DCB *meets the circle at a point equidistant from* B *and* E.

(Fig. 136, Plate X.)—Let the straight line FG bisecting the angles ACE and DCB meet the circle at F; the point F is equidistant from B and E.

Draw FE and BE.

Because $FCBE$ is a quadrilateral, the angles FCB and FEB are together equal to two right angles (III. 22); but the angles FCB and FCA are also equal to two right angles (I. 13), $\therefore ACF = FEB$ (*Ax.* 3); but $ACF = FCE$ (*Hyp.*), $\therefore FCE = FEB$, and the arc FHE = arc FCB (III. 26), and $\therefore$ the chord FE = the chord FB (III. 29).

EXERCISE 137.

If a circle be described about an equilateral triangle, and any point be taken in the circumference, the sum of the lines drawn from that point to the two adjacent angles shall be equal to the line drawn from the same point to the opposite angle.

(Fig. 137, Plate X.)—Let ABC be the equilateral triangle, and D any point in the circumscribing circle; join DA, DB, DC: $DA + DC = DB$.

Produce DC to K, making $CK = AD$ and join BK.

Because $ABCD$ is a quadrilateral inscribed in a circle, $\therefore$ the angles BAD, BCK are equal, and $BA, AD = BC, CK$, $\therefore$ the triangles DBA, KBC are equal (I. 4), whence $DB = BK$, but BDK is the angle of an equilateral triangle; $\therefore$ DBK is an equilateral triangle, and $\therefore$ $DB = DK = DC + DA$.

OTHERWISE, make $DE = DA$ and join AE; the angles ADB, ACB are equal, $\therefore$ ADE is an equilateral triangle, whence $BAC = DAE$; take the common angle away, and the remainders BAE, CAD are equal; $\therefore$ triangles BAE, CAD are equal (I. 4), whence $DC = BE$; $\therefore$ $DA + DC = DB$.

EXERCISE 138.

A is a given point: it is required to draw from A two straight lines which shall contain a given angle and intercept on a given straight line a part of given length.

(Fig. 138, Plate X.)—From BC, the given indefinite straight line, cut off DE equal to the given finite straight line; on DE describe the segment DFE containing an angle equal to the given angle (III. 33); through A draw AF parallel to BC, and meeting the arc in F; join DF and EF, and through A draw AG parallel to FD, and AC parallel to FE: then GC is the portion required, and the angle GAC is equal to the given angle.

Because DF is parallel to GA and EF to CA, the angle DFE = angle GAC; and $DF = GA$, and $EF = CA$ (I. 34); therefore the triangles DFE and GAC are equal in every respect (I. 4); and the base DE = base GC; but DE = the definite straight line, $\therefore$ GC = the definite straight line (*Ax.* 1), and the angle GAC = the given angle, since it is equal to the angle DFE.

EXERCISE 139.

Of all straight lines which can be drawn from two given points to meet in the convex circumference of a given circle, the sum of the two is least, or a minimum, which make equal angles with the tangent at the point of concourse.

(Fig. 139, Plate X.)—Let C and D be two points without the circle PGH, such that the straight line CP and DP make equal angles—viz., CPA and DPB, with AB, the tangent at P: then CP and DP are together less than the sum of any other two straight lines drawn from C and D to any other point in the curve.

Let CG and DG be two other straight lines drawn from C and D to G a point in the curve. Join D and S, where the line CG cuts the tangent AB. DG and GS are together greater than DS (I. 20); to each of these add SC, and DG and GC are greater than DS and SC; but DS and SC are (*Ex.* 149, Part I.) greater than DP and PC, $\therefore$ DG and GC are much greater than DP and PC; hence DP and PC are less than any other two straight lines drawn from D and C to any point in the curve, $\therefore$ they are the least, or a minimum.

EXERCISE 140.

A and B are two fixed points without a circle PQR; it is required to find a point P in the circumference, so that the sum of the squares described on AP and BP may be the least possible.

(Fig. 140, Plate X.)—Bisect the line joining AB, in M; join M with the centre K, and let MK cut the circumference in P; P is the point required.

Join AP, BP.

$AP^2 + BP^2 = 2AM^2 + 2MP^2$ (*Ex.* 255, Part I.); but MP is the least possible line that can be drawn from M to the circumference (III. 8), $\therefore$ $AP^2 + BP^2$ is a minimum.

EXERCISE 141.

If a common tangent be drawn to any number of circles which touch one another internally in the same point; and from any point in this tangent, as centre, a circle be described cutting the others; and from this centre lines be drawn through the intersections of the circles respectively: the segments of them within each circle will be equal.

(Fig. 141, Plate X.)—Let the circles touch each other in the point A; let PB be a common tangent at the point A; from any point P, in the tangent, as centre, let a circle be described cutting the others in C, D, E; from P let the three lines PG, PH, PK be drawn, meeting the circumferences again in G, H, K; then CG, DH, and EK are equal.

Because the rectangles $GP \cdot PC$, $HP \cdot PD$, $KP \cdot PE$ are each equal to AP^2 (III. 36), and that $CP = DP = EP$, $\therefore$ (*Ax.* 3) $CG = DH = EK$.

EXERCISE 142.

Find the centre of a circle such that the tangents drawn to it from three given points A, B, C shall be equal.

(Fig. 142, Plate X.)—Join BA, BC, and from their middle points draw EF, DF at right angles to them respectively meeting at F : F is the required point.

Join FA, FB, FC, and produce them to meet the circumference of any circle, MNS, described with F as centre, and radius less than AF, in R, K, L. Triangles AFE, BFE are equal (I. 4), $\therefore$ $AF = BF$; for a similar reason $CF = BF$, hence $AF = BF = CF$, but $MF = NF = SF$, $\therefore$ $AM = BN = CS$, and $RA \cdot AM = KB \cdot BN = LC \cdot CS$; and hence tangents drawn from the points, A, B, C, to the circle would be equal (III 36).

EXERCISE 143.

Given a straight line and two points on the same side of it; find a point in the given line at which the two given points shall subtend the greatest angle.

(Fig. 143, Plate X.)—Let KS be the given straight line, and A, B the given points. Join AB.

Describe a circle ABL, passing through A, B, and touching KS in L : L is the point required.

Take any other point N; join AL, BL, AN, BN and let BN cut the circumference in M; draw AM.

$AMB = ALB$ (III. 21), but AMB is greater than ANB (I. 16), $\therefore$ ALB is greater than ANB.

EXERCISE 144.

AB is a diameter of a circle; AC and AD are two chords meeting the tangent at B, at E, and F respectively: show that the angles FCE, FDE are equal.

(Fig. 144, Plate X.)—Join BC, BD, and DC. $DBF = DCB$ (III. 32), and $FDB = BCE$ (III. 22 and I. 13), whence $DCE + DFE =$ the angles of triangle $FDB =$ two right angles; $\therefore$ the quadrilateral $FDCE$ is inscribable in a circle, wherefore the angles FCE, FDE are equal (III. 21).

EXERCISE 145.

If one circle touch another internally, any chord of the greater circle which touches the less shall be divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

(**Fig. 145, Plate X.**)—Let *ACBP* and *PEDF* be two circles, of which *PEDF* touches *ACBP* internally in the point *P*, and let *AB* be any chord of the greater touching the less at *D*; then *AD* and *DB* subtend equal angles at the point *P*.

Through *D* draw *PDC*; join *PA*, *PB*, *AC*, *BC*, and *ED*.

Because *PC* cuts off similar segments from each of these circles (*Ex.* 43, Part II.), the angles *DEP* and *CAP* are equal, and the line *AC* is parallel to *ED* (I. 28), ∴ the angle *DAC* = angle *ADE* (I. 29). But the angle *DAC*, *i.e.*, *BAC*, = angle *BPC*, *i.e.*, *BPD*, ∴ the angle *ADE* = angle *BPD* (*Ax.* 1). But the angle *ADE* = the angle *EPD* (III. 32), ∴ the angle *BPD* = angle *EPD*.

EXERCISE 146.

If two circles touch externally, and if a right line be drawn cutting both, and if the four points of intersection be joined with the point of contact, prove that the angle contained by the extreme lines is supplemental to that contained by the intermediate lines.

Fig. 146, Plate X.)—Let the circles *SKM*, *LRM* touch in the point *M*, and *SL* the line cutting them in *S*, *R* and *L*, *R* respectively; join *ML*, *MS*, *MR*, *MK*: then *SML* and *KMR* are supplemental.

Through *M* draw the common tangent *WMG*; then *WMK* = *KSM* (III. 32), and *WMR* = *RLM*, ∴ *KMR* = *KSM* + *RLM*, and is therefore supplemental to *SML*.

EXERCISE 147.

C is the centre of a circle, CP is a perpendicular on a chord APB: show that the sum of CP and AP is greatest when CP is equal to AP.

(**Fig. 147, Plate X.**)—Let *DE* be any other chord; from *C* draw *CF* perpendicular to *DE*; join *AC* and *CD*.

$AC^2 = AP^2 + PC^2 = 2AP^2$ (*Hyp.*); and $DC^2 = DF^2 + FC^2$; but $AC^2 = DC^2$, $\therefore 2AP^2 = DF^2 + FC^2$ (*Ax.* 1): that is, $AP^2 + PB^2 = DF^2 + FC^2$; and $\therefore$ the sum of DF and FC must be less than AB (II. 9), that is, of $AP + PC$. If the sum of DF and FC were equal to AB, their squares would be greater than the squares of AP and PB, or of AP and PC.

EXERCISE 148.

ABC is a right-angled triangle; from any point D in the hypotenuse BC a straight line is drawn at right angles to BC, meeting CA at E and BA produced at F: show that the square on DE is equal to the difference of the rectangles $BD \cdot DC$ and $AE \cdot EC$; and that the square on DF is equal to the sum of the rectangles $BD \cdot DC$ and $AF \cdot FB$.

(Fig. 148, Plate X.)—Join FC.

Because the angles at A and D are right angles, the quadrilateral $AEDB$ is inscribable in a circle (III. 22, *Cor.*); and $\therefore$ the rectangle $BC \cdot CD =$ rectangle $AC \cdot CE$. But rectangle $BC \cdot CD = BD \cdot DC + DC^2$ (II. 3), and $AC \cdot CE = AE \cdot EC + EC^2$; therefore $BD \cdot DC + DC^2 = AE \cdot EC + EC^2 = AE \cdot EC + ED^2 + DC^2$ (I. 47). From these equals take DC^2, and $BD \cdot DC = AE \cdot EC + ED^2$, or $ED^2 = BD \cdot DC - AE \cdot EC$.

Again, because the angles at A and D are right angles, the points A and D are in the circumference of the circle described on the hypotenuse FC (III. 31); and $\therefore$ the rectangle $AE \cdot EC =$ rectangle $FE \cdot ED$ (III. 35), and hence $FE \cdot ED + ED^2 = AE \cdot EC + ED^2$ (*Ax.* 2). But $AE \cdot EC + ED^2 = BD \cdot DC$ (1st part), $\therefore FE \cdot ED + ED^2$ or $DF \cdot ED$ (II. 3) $= BD \cdot DC$. But $DF \cdot FE = BF \cdot FA$; and $DF \cdot FE + DF \cdot ED = FD^2$ (II. 2); $\therefore BF \cdot FA + BD \cdot DC = FD^2$.

EXERCISE 149.

Let $ACDB$ be a semicircle whose diameter is AB; and AD, BC any two chords intersecting in P; then $AB^2 = DA \cdot AP + CB \cdot BP$.

(Fig. 149, Plate X.)—Join DB. ADB is a right angle (III. 31). $DA \cdot AP = DP \cdot PA + PA^2$; (II. 3) and $CB \cdot BP = CP \cdot PB + PB^2$, but $DP \cdot PA = CP \cdot PB$ (III. 35), $\therefore DA \cdot AP + CB \cdot BP = PA^2 + PB^2 + 2DP \cdot PA = AB^2$ (II. 12).

EXERCISE 150.

Two circles ABCD and EBCF, having the common tangents AE and DF cut each other at B and C, and the chord BC is produced to cut the tangents at G and H: show that the square on GH exceeds the square on AE or DF by the square on BC.

(Fig. 150, Plate X.)—$HG^2 = HC^2 + CB^2 + BG^2 + 2HC \cdot CB + 2HC \cdot BG + 2CB \cdot BG$ (II. 4) $= 4BG^2 + 4HC \cdot CB + CB^2$, because $HC = BG$. But $4BG^2 + 4HC \cdot CB = 4BG^2 + 4GB \cdot BC = 4CG \cdot GB$ (II. 3) $= 4AG^2$ (III. 36) $= AE^2$ (II. 4). Therefore $HG^2 = AE^2 + CB^2$ (*Ax.* 1); or HG^2 is greater than AE^2 by CB^2.

EXERCISE 151.

If through any point in the common chord of two circles which intersect, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

(Fig. 151, Plate XI.)—Let any two chords CNQ, DNM be drawn through any point N in the common chord AB of the two intersecting circles ABM, ABQ, one in each circle: C, D, Q, M all lie in the circumference of a circle.

$MN \cdot ND = AN \cdot NB = CN \cdot NQ$ (III. 35); $\therefore$ if a circle be described passing through the three points C, D, Q it will also pass through the fourth M; for if not let it pass, if possible, through any other point, K, then $DN \cdot NK = CN \cdot NQ$, whence $DN \cdot NK = DN \cdot NM$, which is absurd, and $\therefore$ C, D, Q, M all lie in the circumference of a circle.

EXERCISE 152.

If chords be drawn from any fixed point in the circumference of a circle, be cut by another chord which is parallel to the tangent at that point, the rectangle contained by each chord, and the part of it intercepted between the given point and the given chord, is constant.

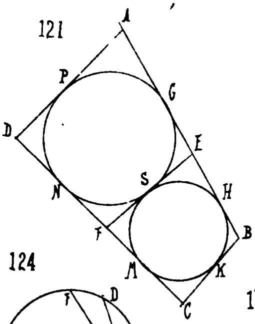
(Fig. 152, Plate XI.)—Let M be the point, EN the tangent, and RQ a chord parallel to the tangent, and let MK, MH be any two chords intercepted by RQ in the points S and F respectively; then $KM \cdot MS = HM \cdot MF$.

Join KH.

Angle EMK = angle MHK (III. 32), but $EMK = MSF$

PLATE IX.

121



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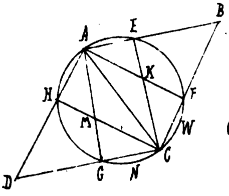

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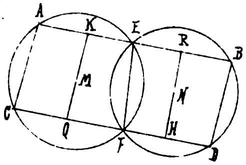

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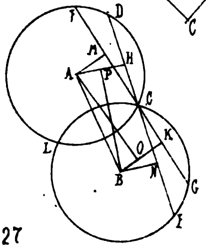

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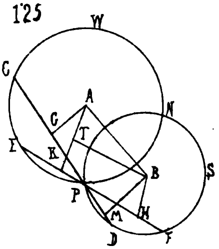

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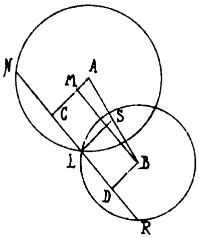

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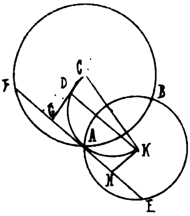

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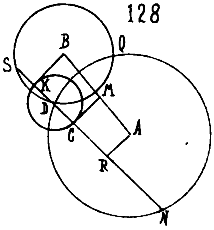

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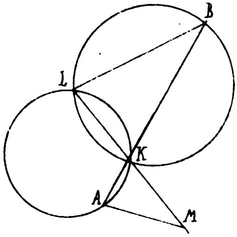

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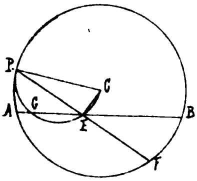

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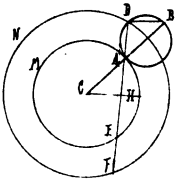

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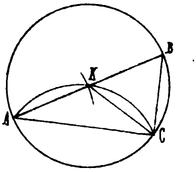

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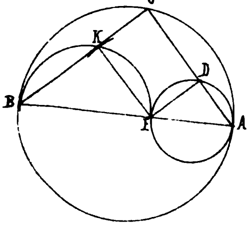

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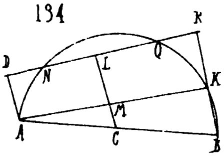

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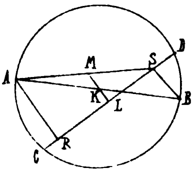

PLATE X.

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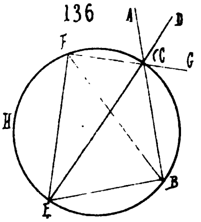

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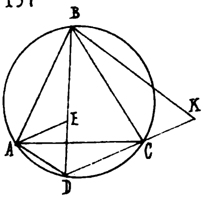

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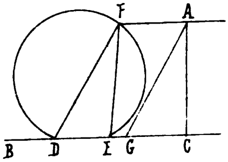

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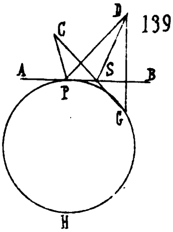

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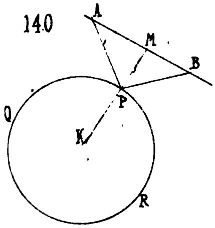

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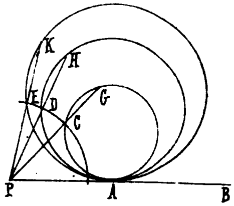

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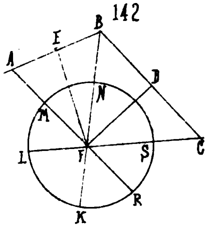

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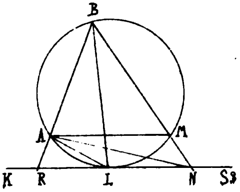

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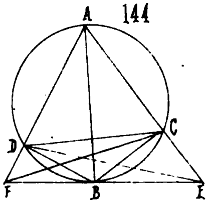

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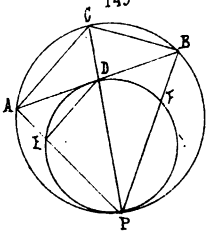

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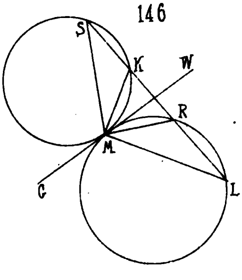

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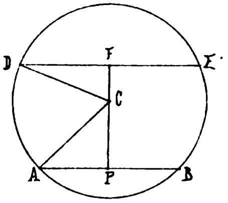

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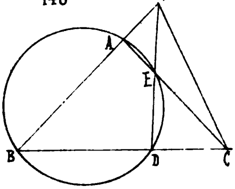

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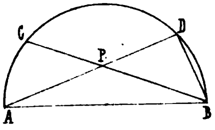

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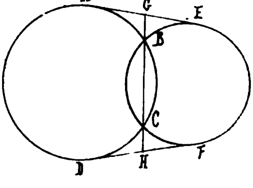


(I. 29), $\therefore MHK = MSF$; $\because MHK + KSF = MSF + KSF$ = two right angles, and $\therefore$ a circle can be described about the quadrilateral $KSFH$ (III. 22); $\therefore KM \cdot MS = HM \cdot MF$ (III. 36).

EXERCISE 153.

If any chord of a circle be produced equally both ways, and tangents to the circle be drawn on opposite sides of it from its extremities, the line joining the points of contact bisects the given chord.

(Fig. 153, Plate XI.)—Let AB be the chord produced both ways so that $AD = BC$, DE and CF tangents on opposite sides of DC, and EF the line joining the points of contact; then EF bisects the chord AB, where it intersects it at M.

Draw CN parallel to DE meeting EF at N, and produce CF to K.

Triangles DEM, CNM are equiangular (I. 15, 29); and $DE = CF$ because $BD \cdot DA = AC \cdot CB$. Again, the angle $KFN = DEM$ (III. 32) $= CNM$, $\therefore CFN = CNF$ and $CN = CF$ (I. 6) $= DE$; $\therefore$ triangles DEM, CNM are equal (I. 26), and $DM = CM$, but since $AD = BC$ $\therefore AM = BN$.

EXERCISE 154.

If from any point in a circular arc perpendiculars be let fall on its bounding radii, the distance of their feet is invariable.

(Fig. 154, Plate XI.)—Let BCD be a circular arc in which C is any point; CK, CM the perpendiculars on the bounding radii AB, AD: then the distance KM is invariable.

Join KM and AC.

The angles at K and M are right, $\therefore AKCM$ is inscribable in a circle whose diameter AC is constant; and since the angle KAM is constant, $\therefore KM$ is invariable (III. 26, 29).

Otherwise—Complete the circle BDQ, and produce CK, CM to meet the circumference in H, L respectively; join HL. Now, because the angles at A, K, M are constant, $\therefore$ the angle at C and the arc HQL is constant, hence the chord HL is constant, and its half KM is invariable.

EXERCISE 155.

From a given point M as centre describe a circle cutting a straight line SZ in two points, so that the rectangle contained by their distances from a given point R in SZ may be equal to a given square.

(Fig. 155, Plate XI.)—Join MR, and on it describe the semicircle MNR; with R as centre, and a side of given square as distance, describe an arc cutting the semicircle in N; join NR, NM, and with MN as radius describe the circle NKL cutting SZ in L and K: L and K are the points required.

For since MN is a radius, and the angle at N a right angle, $LR \cdot KR = RN^2$ (III. 36).

EXERCISE 156.

The diameter ACD of a circle, whose centre is C, is produced to P, determine a point F in the line AP such that the rectangle $PF \cdot PC$ may be equal to the rectangle $PD \cdot PA$.

(Fig. 156, Plate XI.)—Draw the tangent PK and join CK; with CK as diameter describe the circle CFK cutting CP in F: F is the point required.

Now, since PK is a tangent to both circles, $PF \cdot PC = PD \cdot PA$ (III. 36).

EXERCISE 157.

In the diameter produced, of a circle, determine a point so that the tangent drawn from it to the circumference shall be of given length.

(Fig. 157, Plate XI.)—Let the diameter AB be produced in the direction AB *ad infinitum*; draw BR at right angles to AB and equal to the given length; join R with the centre of the circle K; make $KM = KR$: M is the point required.

From M draw the tangent MN (III. 17), and join KN.

The sides MK, $KN = RK$, KB, and the angles at N and B are right, $\therefore MN = BR$ (I. 47).

EXERCISE 158.

Through a given point which is outside a circle draw a line cutting the circle, so that the rectangle under the whole line and external segment shall be equal to the square of the intercepted chord.

(Fig. 158, Plate XI.)—Let P be the point.

From the point P draw the tangent PM (III. 17), and through P draw PKN so that KN may be equal to PM (*Ex.* 42): $NP \cdot PK = MP^2$ (III. 36) $= KN^2$.

EXERCISE 159.

From a given point outside a circle draw a secant such that the sum of the squares of the secant and external segment shall be given.

(Fig. 159, Plate XI.)—Let P be the given point, and DFE the given circle; it is required to draw a secant from P such that the sum of the squares on the whole secant and the part outside the circle shall be equal to a given square.

From P draw the tangent PB (III. 17); from the given square take twice the square on PB; and from P draw the secant PAC so that the part inside the circle may be equal to one side of the difference—viz., of twice PB^2 and the given square (*Ex.* 42): then PC is the secant required.

$CP^2 + PA^2 = 2CP \cdot PA + AC^2$ (II. 7); but $2CP \cdot PA = 2PB^2$ (III. 36); and $\therefore CP^2 + PA^2 = 2PB^2 + AC^2 =$ the given square; and $\therefore PC$ is the secant required.

EXERCISE 160.

If from the extremities of the diameter of a circle tangents be drawn; any other tangent to the circle, terminated by them, is so divided at the point of contact that the rectangle of its segments equals the square of the radius.

(Fig. 160, Plate XI.—Let MS, KL be tangents from the extremities of the diameter MK, and SRL any other tangent touching the circle in R and the tangents in the points S, L respectively: $LR \cdot RS = CR^2$.

Join CS, CL.

Triangles SMC, SRC are equal (I. 8), $\therefore$ angle $SCM =$ angle SCR; for a similar reason angle $RCL =$ angle KCL, $\therefore$ the whole angle $SCL =$ the sum of the angles SCM, KCL; but the sum of the angles SCL, SCM, $KCL =$ two right angles, $\therefore SCL$ is a right angle, whence SL is the diameter of a circle passing through C, and CR is at right angles to it, $\therefore LR \cdot RS = CR^2$ (III. 35).

EXERCISE 161.

Given a point O outside a circle find another P in the line joining O with the centre, such that if a tangent PN be drawn to the circle the line PN shall be equal to OP.

(Fig. 161, Plate XI.)—Let M be the centre of the given circle, and let the circumference cut OM in S; cut OM in K making $OM \cdot MK = MS^2$, and bisect OK in P: P is the point required.

On OK describe a semicircle cutting the circumference of the given circle in N, and join MN, PN.

$OM \cdot MK = MS^2 = MN^2$, $\therefore$ MN is a tangent to KNO (III. 37); hence PN is also a tangent to SNW and equals OP.

Otherwise—Draw the radius MW at right angles to OM, and join OW cutting the circumference at N; draw NP cutting OM in P, and making angle $PNO = PON$; join MN: then P is the point required.

For the angles of triangle WMO equal the angles at N on one side of OW, but since $MNW = MWN$, and $PNO = PON$, $MNP = OMW =$ a right angle, $\therefore$ PN is a tangent (III. 16), and because the angles PNO, PON are equal, $PO = PN$.

EXERCISE 162.

Describe a circle which shall pass through two given points, and bisect the circumference of a given circle.

(Fig. 162, Plate XI.)—Let A, B be the given points and MLS the given circle, the centre of which is C.

Draw AC, cutting the circumference of MLS in M, and produce it to N making $AC \cdot CN = CM^2$; describe the circle BAN cutting the given circle in R, S: BAN is the circle required.

Join RC.

RC produced will pass through S, for, if possible, let RC produced meet MLS, BAN in L, K respectively.

Because RCK, NCA are chords of BAN intersecting in C, $\therefore$ $AC \cdot CN = RC \cdot CK$ (III. 35), but $AC \cdot CN = CM^2 = RC \cdot CS$, $\therefore$ $RC \cdot CS = RC \cdot CK$, and $\therefore$ CS or $CL = CK$, which is impossible, hence BAN bisects the circumference of the given circle.

EXERCISE 163.

Describe a circle which shall pass through a given point and bisect the circumferences of two given circles.

(Fig. 163, Plate XI.)—Let Z and L be the centres of the circles SBR and KQV respectively, and P the given point.

Join PZ, PL and produce them to N and M respectively, making the rectangle under the segments of PN equal to the square of the radius of SBR, and the rectangle under the segments of PM equal to the square of the radius of KQV: the circle NMP cuts the circumference of the circles in the manner required. (See *Ex.* 162).

EXERCISE 164.

Describe a circle which shall bisect the circumferences of three given circles.

(Fig. 164, Plate XI.)—Let AKC, nmh, DEF be the given circles, and M, N, S their respective centres; join MS, MN, and draw the radii MG, SH at right angles to MS; join GH, and from Z, its middle point, draw ZQ at right angles to it, meeting MS in Q, and with Q as centre describe a semicircle passing through the points G, H, and meeting MS produced in a, T.

Now, because aGT is a semicircle, and GM a perpendicular on the diameter, $\therefore$ $TM \cdot Ma = MG^2$; for a similar reason $AS \cdot ST = HS^2$.

In like manner produce MN both ways to V and L making $LM \cdot MV = MG^2$, and $VN \cdot NL =$ the square of the radius of nmh.

Because $TM \cdot Ma = LM \cdot MV$, a circle can be described passing through $VaLT$, which shall be the circle required. (*Vide Ex.* 162).

EXERCISE 165.

Describe a triangle each of whose sides shall pass through a given point, and such that it shall be equilateral with a given triangle.

(Fig. 165, Plate XI.)—Let W, H, S be the points.

Join SW, SH, and on these describe the segments SKW, SMH, so that the segment on SW may contain an angle equal to one of the angles of the given triangle, and that on SH an angle equal to another, of the same triangle; through *S* draw *KSM* touching the arcs in *K*, *M*, and equal to the side adjacent to the aforesaid angles of the given triangie (*Ex.* 127).

Join *KW*, *MH*, and produce them to meet in *L*: *KLM* is the triangle required (I. 26).

EXERCISE 166.

Describe a triangle in a circle, two of whose sides shall pass each thrcugh a given point, and whose third side shall be parallel to the line joining these points.

(Fig. 166, Plate XII.)—Let *KCT* be the given circle, and *A*, *B* the given points.

Join *AB*, and cut it at *S*, making $AB \cdot BS = BT^2$, a tangent drawn to the circle from *B*; from *S* draw the tangent *SL* touching the circle in *L*; join *BL* and produce it to meet the circle at *C*, and draw *AC* cutting the circle at *K*; join *KL*: *KCL* is the triangle required.

Because $AB \cdot BS = BT^2$, and $CB \cdot BL = BT^2$, $\therefore AB \cdot BS = CB \cdot BL$, and hence *ACLS* is circumscribable by a circle, $\therefore$ $LSB = LCA$, but since *SL* is a tangent, $SLK = LCA$ (III. 32); $\therefore LSB = SLK$, hence *KL* is parallel to *AB* (I. 28), and *KCL* is the triangle required. By drawing another tangent from *S*, and producing as before, we can have another triangle satisfying the conditions.

EXERCISE 167. (*Ex.* 166, SECOND CASE.)

Let the points be within the circle.

(Fig. 167, Plate XII.)—Produce *AB* both ways to meet the circle at *R*, *S*, and also produce it to *T*, making $RB \cdot BS = AB \cdot BT$. Now, because *RB* is greater than *AB*, the point *T* must be outside the circle; from *T* draw the tangent *TL*, and produce it to *M*; join *LB*, and produce it to meet the circle at *C*, and let *CA* produced meet the circle at *K*; join *KL*: *KCL* is the triangle required.

Because $CB \cdot BL = RB \cdot BS = AB \cdot BT$, the points *A*, *C*, *T*, *L* are in the circumference of a circle, $\therefore LTB = LCK$, but *MLK* also equals *LCK* (III. 32), $\therefore LTB = MLK$, and hence *KL* is parallel to *AB* (I. 28), $\therefore$ *KCL* is the triangle required.

If *AB* be produced in the direction *AR*, and a similar construction effected, it will be found that another triangle fulfils the necessary conditions.

EXERCISE 168.

Describe a triangle in a circle, two of whose sides shall pass each other through a given point, and whose third side shall be parallel to a given line.

(Fig. 168, Plate XII.)—Let the line *RH* joining the points through which the sides must pass, and the line *KR* to which the base must be parallel, intersect in the point *R*. Take any point *Q*, in the circumference of the given circle, and draw *QW*, *QX* making the angle *WQX* = *HRK*, and join *WX*.

Again, cut *RH* in *S*, making *RH·HS* equal the square of the tangent from *H* to the circle; from *S* draw *SND*, cutting the circle at *N* and *D*, making *ND* = *WX* (*Ex.* 42); join *HN*, and produce it to meet the circle again at *M*; join *MR* cutting the circle in *L*, and join *LN*: *LMN* is the triangle required.

Join *LD*.

Now, because *RH·HS* = *MH·HN*, *RMNS* is inscribable in a circle, and *NSH* = *LMN* = *LDN*, ∴ *LD* is parallel to *RH* (I. 27), but since *ND* = *WX*, *NLD* = *WQX* (III. 28, 27) = *HRK*, ∴ *MLN* = *MRK*, hence *LN* is parallel to *KR* (I. 28), ∴ *LMN* is the triangle required.

EXERCISE 169.

Describe a triangle in a circle, each of whose sides shall pass through a given point.

(Fig. 169, Plate XII.)—Let *EDFG* be a given circle, and *A*, *B*, *C* three given points: it is required to describe a triangle in the circle *EDFG*, so that its sides shall pass through *A*, *B*, *C*.

Make the rectangle *AB·BH* equal to the square of the tangent from *B* to the given circle *EDFG*. Inscribe a triangle *EFG* in the given circle, having its two sides *EG* and *FG* passing through the fixed points *C* and *H*, and its base *EF* parallel to the given straight line *AB* (*Ex.* 168). Join *BG*, and produce it to meet the circumference again in *D*; join *DE*; then *DEG* is the required triangle.

For, if possible, let *DE* produced not pass through *A*; and let it meet *AB* in *K*.

Because *EF* is parallel to *AB*, the angle *EFG* = angle *GHB* (I. 29); and the angle *EFG* = angle *EDG* (III. 21); and ∴ the angle *GHB* = angle *EDG* (*Ax.* 1). But the angles

GHB and GHK = two right angles, ∴ the angles EDG and GHK = two right angles, and the quadrilateral $DEKHG$ is inscribable in a circle (III. 32, *Cor.*); and ∴ the rectangle $DB \cdot BG$ = rectangle $KB \cdot BH$. But the rectangle $DB \cdot BG$ = rectangle $AB \cdot BH$ (*Cons.*), since each = the square of the tangent from B. Hence the rectangle $AB \cdot BH$ = rectangle $KB \cdot BH$, which is impossible; and ∴ the triangle EDG is the required inscribed triangle. Another triangle, fulfilling the required conditions, can also be inscribed in the given circle; for two triangles can be inscribed, having their sides passing through H, C and their third side parallel to AB.

EXERCISE 170.

Given in position three points A, B, C; through one of them, C, draw a right line so that if from the remaining two the perpendiculars AP, BQ be let fall upon it: $PC + CQ$ shall be given.

(Fig. 170, Plate XII.)—Join CA, CB, and on them describe the semicircles CPA, CQB respectively; through C draw PCQ equal to the given line and meeting the semicircles in P, Q (*Ex.* 127): PCQ is the line required.

Join AP, BQ.

The angles at P, Q are right angles (III. 31), ∴ PCQ is the line required.

EXERCISE 171.

Given in position three points A, B, C; through one of them, C, draw a right line so that if from the remaining two the perpendiculars AP and BQ be let fall upon it: $PC - CQ$ shall be given.

(Fig. 171, Plate XII.)—Describe the semicircle BKA on the line joining AB, and in it place BK equal to the given line; join AK, and draw CP parallel to BK meeting AK in P: CP is the line required.

Let fall the perpendicular BQ.

Because the angles at K and Q are right angles, and CP parallel to BK, ∴ $BKPQ$ is a rectangle, hence $QP = BK$ (I. 34), ∴ $PC - CQ = QP = BK$.

EXERCISE 172.

Given in position three points A, B, C; through one of them, C, *draw a right line so that if from the remaining two the perpendiculars AP and BQ be let fall upon it : PC·CQ shall be given.*

(FIRST CASE).

(Fig. 172, Plate XII.)—Join CA, CB; on CB describe the semicircle CSB, cutting the circle CSA described, with CA as diameter, in S; join CS and cut it in H making $SC \cdot CH$ equal to the given quantity; draw HQ cutting the arc CSB in Q and making the angle CHQ equal to the angle in the segment CPS; join CQ and produce it to meet the circumference of CSA in P: CP is the line required.

Join BQ, AP, SP.

$CHQ = CPS$ (*Cons.*), $\therefore$ $HQPS$ is inscribable in a circle, hence $PC \cdot CQ = SC \cdot CH$, or the given quantity, and AP, BQ are perpendiculars (III. 31).

EXERCISE 173. (*Ex.* 172, SECOND CASE).

(Fig. 173, Plate XII.)—If the given quantity be greater than CS^2, produce CS to H making $SC \cdot CH$ equal to the given quantity, and draw HP making CHP equal to the angle in the segment CQS, and cutting CSA in P; draw CP cutting the arc CSB in Q: CP is the line required.

Join BQ, AP, SQ.

$CQS = CHP$ (*Cons.*), $\therefore$ $QPHS$ is inscribable in a circle, hence $PC \cdot CQ = SC \cdot CH$ (III. 36), or the given quantity, and AP, BQ are perpendiculars (III. 31).

EXERCISE 174.

Given in position three points A, B, C; through one of them, C, draw a right line so that if from the remaining two the perpendiculars AP and BQ be let fall upon it, AP + BQ shall be given—that is, equal to a given line.

(Fig. 174, Plate XII.)—Let A, B, C be the three given points.

Join AC, and produce it, making CD equal to AC; join BD, and on it describe a semicircle, in which place BE equal to the given line; join DE; through C draw QP parallel to DE (I. 31); from D and A draw DH and AP perpendicular to QP.

Because the angles at P and H are right angles (*Cons.*), the angle HCD = angle ACP (I. 15), and $AC=CD$, $\therefore AP=HD$. Again, because the angle DEB is a right angle (III. 31), and QP is parallel to DE, the angle BQP is also a right angle (I. 29), and $QHDE$ is a rectangle, and $QE = DH = AP$; and, therefore, $BE = BQ + QE = BQ + DH = BQ + AP$.

EXERCISE 175.

If two chords intersect at right angles within a circle the sum of the squares described on the four segments equals the square described upon the diameter.

(**Fig. 175, Plate XII.**)—Let the chords AB, CD intersect at right angles at the point E: $AE^2 + EB^2 + CE^2 + ED^2$ = the square of the diameter BL.

Draw DB, DL, DA, and AC.

Because LB is a diameter, the angle LDB is right, and $\therefore$ equal to AED: also the angle at $L = BAD$ (III. 21), $\therefore$ $LBD = ADE$ (I. 32), whence $LD = AC$ (III. 26, 29), but $LB^2 = LD^2 + DB^2 = AE^2 + CE^2 + DE^2 + BE^2$.

EXERCISE 176.

If two chords of a circle intersect at right angles without a circle, the sum of the squares of the four segments shall be equal to the square of the diameter.

(**Fig. 176, Plate XII.**)—Let AB, CB be the chords intersecting at B, and cutting the circle in the points M and N respectively: $AB^2 + CB^2 + BM^2 + BN^2 = (\text{diameter})^2$.

Draw MK parallel to CB, meeting the circle in K, and join KA, KC, and MN.

Because the angle at M is right, AK is a diameter.

$AK^2 = AC^2 + KC^2$, but since BC is parallel to MK, $\therefore$ $KC = MN$; and $AC^2 = AB^2 + BC^2$, $\therefore AK^2 = AB^2 + BC^2 + BM^2 + BN^2$.

EXERCISE 177.

Through a point inside a circle draw a line so that the sum of the squares of the segments may be given.

(**Fig. 177, Plate XII.**)—Through M the given point draw AMD, a chord of the given circle, and such that AD^2 may be equal to the given square, together with twice the rectangle under the segments of any chord of the given circle drawn through the given point: then AD is the line required.

For $AD^2 = 2AM \cdot MD + AM^2 + MD^2$ (**II. 4**), but $AD^2 = 2AM \cdot MD$, together with the square of the given line (*Cons.*), $\therefore AM^2 + MD^2$ equal the square of the given line.

EXERCISE 178.

Draw a chord of a circle parallel to a diameter which shall subtend a right angle at a given point in this diameter.

(Fig. 178, Plate XII.)—Find $LN^2 = RM^2 + SM^2$, the segments into which the diameter RS is divided at M, and describe the concentric circle WHB, touching a chord of the given circle equal to LN (*Ex.* 42); draw LWN, touching WHB, parallel to RS, and meeting the outer circle at L, N: LN is the chord required.

Draw ML, MN.

$LN^2 = RM^2 + SM^2 = ML^2 + MN^2$ (*Ex.* 30); $\therefore LMN$ is a right angle (**I. 48**).

EXERCISE 179.

Through a point given in position within the legs of an angle, draw a line so that the rectangle under the segments of this line shall be given.

(Fig. 179, Plate XII.)—Let LSR be the given angle and M the given point.

Join SM and produce it to H, making $SM \cdot MH$ equal to the given rectangle, and on MH describe the segment MRH containing an angle equal to LSH (**III. 33**), cutting SR in R; join RM and produce it to cut SL in L : RL is the line required.

Draw HR.

Because angle $LRH = LSH$ (*Cons.*) $\therefore$ the points L, S, R, H, are in the circumference of a circle, $\therefore RM \cdot ML$ equals $SM \cdot MH$ (**III. 35**), or the given rectangle.

EXERCISE 180.

Through a point given in position without the legs of an angle, draw a line so that the rectangle under the segments of this line intercepted between the given point and the lines containing the angle shall be given.

(Fig. 180, Plate XII.)—Let RSL be the given angle and M the given point.

Join SM, and cut it in H, making $SM \cdot MH$ equal to the given rectangle; on *MH* describe the segment *MRH* containing an angle equal to *MSL*, and cutting *SR* in *R*; join *MR*, and produce it to meet *SL* in *L*: *ML* is the line required.

Join *RH*.

Because the angles *MRH*, *MSL* are equal (*Cons.*), ∴ *RLSH* is inscribable in a circle, and hence $LM \cdot MR$ equals $SM \cdot MH$ (III. 36) or the given rectangle.

EXERCISE 181.

If the centres of two circles which touch each other externally be fixed, the common tangent of those circles will touch another of which the straight line joining the fixed centres is the diameter.

(Fig. 181, Plate XIII.)—Let the two circles *AFM* and *BFH* be two circles touching each other at *F*; let *AB* be a common tangent at the points *A* and *B*; let *C* and *K* be the centres of these circles; then the circle described on the line *CK*, joining their centres, will touch the common tangent, or *AB* will also be a tangent to the circle described on *CK*.

From *F* draw the common tangent *FE*, meeting *AB* in *E*; from *E* draw *ED* perpendicular to *AB*, and meeting *CK* in *D*; then *D* is the centre of the circle, and also the middle point of *CK*.

Because $AE = EF = EB$ (*Cor.* III. 17), and *AC*, *ED*, and *BK* are parallel, ∴ $CD = DK$. Again, because the angle AEC = angle CEF (I. 8), and angle ACE = angle DCE, and angle ACE = angle CED (I. 29), the angle ECD = angle CED, and $CD = DE$, and ∴ $DK = DE$ (*Ax.* 1). Hence *D* is the centre of the required circle.

EXERCISE 182.

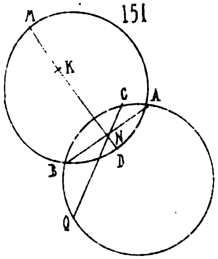
A straight line and two circles are given: find the point in the straight line from which the tangents drawn to the circles are of equal length.

(Fig. 182, Plate XIII.)—Join the centres *M*, *N* of the given circles *SQH*, *KOZ*, and divide *NW* in *R*, making $MR^2 - NR^2 = MS^2 - NK^2$.

Take the point *R* nearer to the centre of the smaller circle, and draw *RC* at right angles to *MN* cutting the given line *AB* in *C*: *C* is the point required.

Join *CM*, *CN*, and draw the tangents *CS*, *CK* to the circles *SQH*, *KOZ* respectively; let $NL^2 = MS^2 - NK^2$.

PLATE XI.

151


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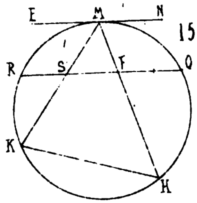

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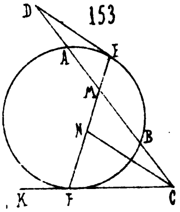

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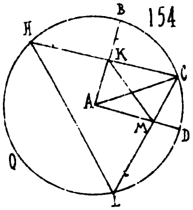

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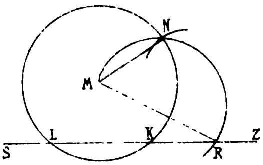

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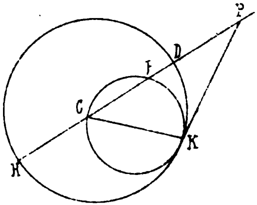

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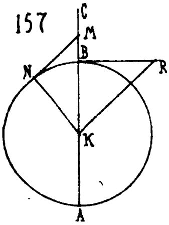

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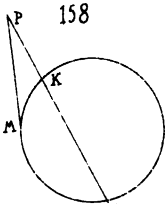

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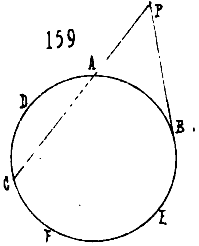

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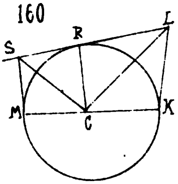

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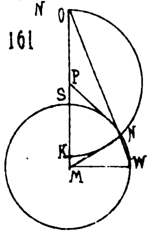

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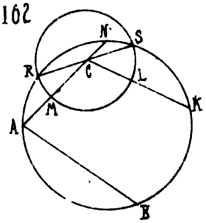

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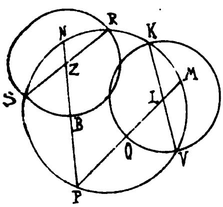

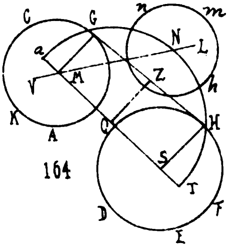

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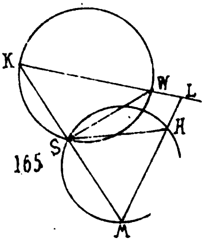
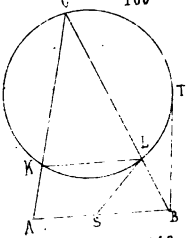


165

PLATE XII.

166



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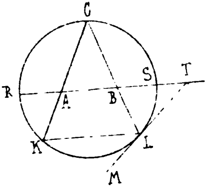

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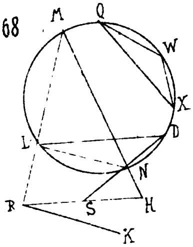

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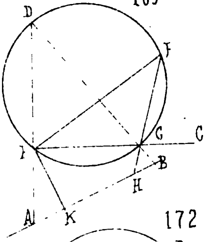

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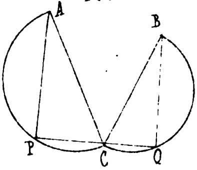

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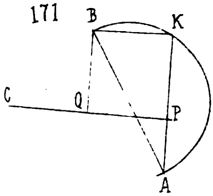

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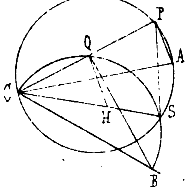

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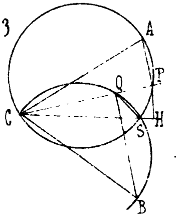

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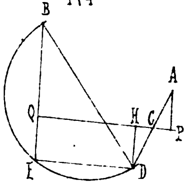

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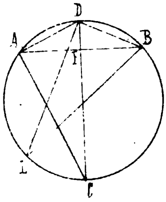

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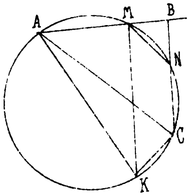

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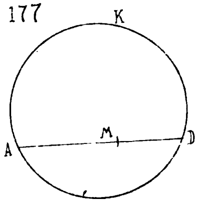

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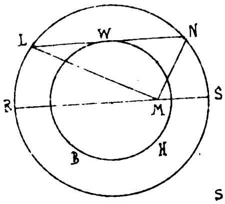

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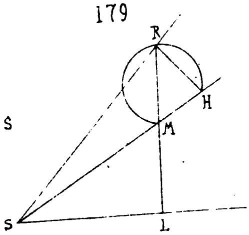

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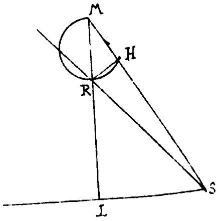


In the triangle MCN, because CR is perpendicular to MN, $MR^2 - NR^2 = CM^2 - CN^2$, but $MR^2 - NR^2 = MS^2 - NK^2 = NL^2$; $\therefore CM^2 = CN^2 + NL^2 = NL^2 + NK^2 + CK^2$. Again, $CM^2 = MS^2 + CS^2$, and $\therefore MS^2 + CS^2 = NL^2 + NK^2 + CK^2$, but $MS^2 = NK^2 + NL^2$; hence $CS^2 = CK^2$, and $CS = CK$.

EXERCISE 183.

If two tangents and a secant be drawn to a circle from any point without it, and the points of contact of the tangents be joined, the rectangle contained by the whole secant and its middle segment is equal to the rectangle contained by its extreme segments.

(Fig. 183, Plate XIII.)—Let PA and PB be tangents from the point P to the circle $ACBE$; let AB be the line joining the points of contact; and let $PCDE$ be any secant from the point P; then the rectangle $EP \cdot DC$ is equal to the rectangle $ED \cdot PC$.

The triangle APB is isosceles (III. 36), and $\therefore AP^2 = DP^2 + AD \cdot DB$ (*Ex.* 253, PART I.); but $AP^2 = EP \cdot PC$ (III. 36), and $AD \cdot DB = ED \cdot DC$ (III. 35); and $\therefore EP \cdot PC = DP^2 + ED \cdot DC$. Again, $DP^2 = DP \cdot DC + DP \cdot PC$ (II. 2), $\therefore EP \cdot PC = ED \cdot DC + DP \cdot DC + DP \cdot PC$ (*Ax.* 1); but $ED \cdot DC + DP \cdot DC = (ED + DP) \cdot DC = EP \cdot DC$; and hence $EP \cdot PC = EP \cdot DC + DP \cdot PC$.

But $EP \cdot PC = ED \cdot PC + DP \cdot PC$; and $\therefore$ (*Ax.* 1) $ED \cdot PC + DP \cdot PC = EP \cdot DC + DP \cdot PC$. Take $DP \cdot PC$ from these last equals, and $ED \cdot PC = EP \cdot DC$.

EXERCISE 184.

Between two lines given in position to draw a line equal to a given line, so that the triangle thus formed may be equal to a given rectilineal figure.

(Fig. 184, Plate XIII.)—Let BA and BC be the given lines, and DE the line whose magnitude is given; it is required to draw a straight line between BA and BC equal to DE, and forming with them a triangle equal to a given rectilineal figure.

On DE describe a segment of a circle containing an angle equal to the angle at B (III. 33); bisect DE in H, and on DH describe a rectangle equal to the given rectilineal figure

(**I. 45**), cutting the circle in K; join KD and KE; cut off from BA a part BM equal to KD, and from BC a part BN equal to KE, and join MN: then MN is the line required.

Because DE is double of DH, the triangle DKE = the rectangle $DSFH$ (**I. 41**, *Cor.*) = the given rectilineal figure (*Cons.*); and DK and KE and the contained angle are respectively equal to MB and BN and the contained angle, ∴ (**I. 4**) $MN = DE$, and the triangle MBN = triangle DKE = the rectangle $DSFH$ = the given rectilineal figure.

EXERCISE 185.

The rectangle under the sides of a triangle is equal to the square on the bisector of the vertical angle together with the rectangle under the segments of the base made by the bisector of the vertical angle.

(**Fig. 185, Plate XIII.**)—Let ABC be the given triangle, and let BD the bisector of the vertical angle be produced to meet the circumscribed circle at F.

From AB cut off $BE = BC$, and join ED.

Triangles DBE, DBC are equal (**I. 4**), ∴ the angles DEB, DCB are equal, but $DCB = AFB$, being in the same segment; ∴ $DEB = AFB$, and hence $AEDF$ is inscribable in a circle, ∴ $AB \cdot BE = FB \cdot BD$; but $BE = BC$, and $FB \cdot BD = FD \cdot DB + DB^2 = AD \cdot DC + DB^2$, ∴ $AB \cdot BC = AD \cdot DC + DB^2$.

EXERCISE 186.

If a triangle ABC be inscribed in a circle, and if lines CD and CE be drawn to meet the side AB in D, and the circle in E, and such that the angles ACE and DCB shall be equal, prove that the rectangle $AC \cdot BC = EC \cdot CD$.

(**Fig. 186, Plate XIII.**)—From AC cut off CK equal to CB; and from EC cut off CF equal to CD, and join KF.

The triangle KCF = triangle BCD (**I. 4**) in every respect, and the angle CKF = angle CBD; but $CBD = CEA$ (**III. 21**); and ∴ the angle CKF = angle CEA (*Ax.* 1). To each add angle AKF, and the angles FEA and FKA = angles CKF and FKA, hence the figure $EAKF$ is inscribable in a circle (**III. 22**, *Cor.*); and ∴ the rectangle $AC \cdot CK$ = rectangle $EC \cdot CF$; but $AC \cdot CK = AC \cdot CB$ (*Cons.*), and $EC \cdot CF = EC \cdot CD$; and therefore, $AC \cdot CB = EC \cdot CD$ (*Ax.* 1).

EXERCISE 187.

The rectangle under the sides of a triangle is equal to the rectangle under the perpendicular to the base and the diameter of the circumscribing circle.

(Fig. 187, Plate XIII.)—Let ABC be a triangle inscribed in a circle ABM; BM, the diameter of the circle, and BD the perpendicular on the base AC: then $AB \cdot BC = MB \cdot BD$.

On AB, MB take BH, BL equal to BC, BD respectively; join HL, AM.

In the triangles BMA, BCD, the angles at A and D are right angles; the angles at M and C are also equal (III. 21), $\therefore$ the remaining angles at B are equal (I. 32), and hence the triangles BLH, BDC are equal (I. 4), $\therefore$ angle $BHL = BCD = BMA$, hence $AHLM$ is circumscribable by a circle, $\therefore$ $AB \cdot BH = MB \cdot BL$, but $BH = BC$, and $BL = BD$, $\therefore$ $AB \cdot BC = MB \cdot BD$.

EXERCISE 188.

If an arc of a circle be cut equally and unequally, prove that the rectangle under the chords of the unequal segments, together with the square of the chord of the intermediate arc, is equal to the square of the chord of half the arc.

(Fig. 188, Plate XIII.)—Let the arc AEB of the circle $AMBE$ be bisected in E, and divided unequally in F; then the rectangle $AF \cdot FB$ together with EF^2 will be equal to AE^2.

Join AB, AE, EF, EB, FB, AF; draw the diameter EM, cutting AB at C, and draw FD; and from F draw FH parallel to AB, and cutting EM in N.

Because the arc AEB is bisected in E, and its chord AB bisected perpendicularly by EM, then EM passes through the centre of the circle (III. 1); and since FH is parallel to AB it is also bisected in N. Now, in the triangle AFB, since FD is perpendicular to AB, the rectangle $AF \cdot FB = ME \cdot FD$ (*Ex.* 187); for the same reason $AE \cdot EB = AE^2 = ME \cdot EC = ME \cdot EN + ME \cdot NC = ME \cdot EN + ME \cdot FD$. But $ME \cdot EN = HE \cdot EF = EF^2$; and $\therefore$ $AE^2 = AF \cdot FB + EF^2$ (*Ax.* 1).

EXERCISE 189.

Given the base, vertical angle, and the sum of the squares of the sides of a triangle: to construct it.

(Fig. 189, Plate XIII.)—Let AB be the given base. On it describe a segment of a circle containing an angle equal to the given vertical angle (III. 33); bisect AB in C; find a line equal to half the difference between the sum of the squares of the sides and twice the square of half AB (*Ex.*); from C as centre, with this line as distance, describe an arc cutting the circumference of the segment in D; join AD and DB; then ADB is the triangle required.

Because AB is bisected in C, and CD joined, $AD^2 + DB^2 = 2AC^2 + 2CD^2$ (*Ex.* 255, PART I.).

EXERCISE 190.

Given the base, vertical angle, and the difference of the squares of the sides of a triangle: to construct it.

(Fig. 190, Plate XIII.)—Let AB be the given base. On it describe a segment of a circle containing an angle equal to the given vertical angle (III. 33); divide AB in D so that the difference of the squares of AD and DB may be equal to the given difference (I. 47, *Cor.*); from D draw DC at right angles to AB; join AC and BC; then ACB is the triangle required.

The angle ACB is (*Cons.*) the vertical angle; AB is the given base; and the difference between the squares of the sides is equal to the difference between the squares of the segments of the base AB; or $AC^2 - CB^2 = AD^2 - DB^2$.

EXERCISE 191.

Given the base, vertical angle, and rectangle of the sides of a triangle: to construct it.

(Fig. 191, Plate XIII.)—Let AB be the given base. On it describe a segment of a circle containing an angle equal to the given vertical angle; complete the circle; then the rectangle of the sides is equal to the rectangle of the diameter of the circumscribing circle and the perpendicular from the vertex on the given base (*Ex.* 187); and, therefore, as the diameter is known, the perpendicular is found. Now, from B raise a perpendicular BG equal to that found; from G draw GD parallel to AB (I. 31), and meeting the circumference in D; join AD and DB, and from D draw DH perpendicular to AB; then it is evident from the construction that ADB is the triangle required.

Since DH is perpendicular to AB, and DCE any diameter, the rectangle $AD \cdot DB$ = rectangle $ED \cdot DH$; and ADB = the vertical angle; and $DH = BG$.

EXERCISE 192.

If from any point in the circumference of the circle described about a triangle perpendiculars be drawn to the sides of the triangle, the three points of intersection shall be in the same straight line.

(Fig. 192, Plate XIII.)—Let ABC be a triangle, and P any point in the circumference of the circumscribing circle; then if from P perpendiculars PD, PE, and PF be drawn to the sides of the triangle, the three points D, E, and F will be in the same straight line.

Because the angles ADP and AEP are right angles (*Cons.*), $\therefore$ a circle can be described about the quadrilateral $AEPD$ (III. 22, *Cor.*), and hence the angle DAP = the angle DEP (III. 21). But DAP and PAC are together equal to two right angles (I. 13), $\therefore$ DEP and PAC are together equal to two right angles. Again, PAC and PBC are equal to two right angles (III. 22), $\therefore$ $DEP = PBC$. Now, since PEB and PFB are right angles (*Cons.*), $PEFB$ is circumscribable by a circle, and the angles PBF and PEF = two right angles; $\therefore$ DEP and PEF are = to two right angles, and DEF is a straight line.

EXERCISE 193.

If from the middle point of a circle which lies above the base of an inscribed triangle, perpendiculars be let fall on the sides, prove that the parts cut off on each side, measured from the vertex, are each equal to half the difference of the sides; and that the parts cut off, measured from the extremities of the base, are each equal to half the sum of the sides.

(Fig. 193, Plate XIII.)—The rectangle $AD \cdot DC$ = rectangle of DE and the diameter of the circle; and the rectangle $BD \cdot DC$ = rectangle of DF and the diameter of the circle; but $AD = BD$ (III. 29), and hence $AD \cdot DC = BD \cdot DC$; and $\therefore$ the rectangle of DE and the diameter of the circle = the rectangle of DF and the diameter of the circle; and, therefore, $DE = DF$. Now, $AD^2 = AE^2 + DE^2$ (I. 47), and $BD^2 = BF^2 + DF^2$; and as $AD^2 = BD^2$, and $DE^2 = DF^2$, $\therefore$ $AE^2 = BF^2$, and $AE = BF$.

Again, $DC^2 = DE^2 + EC^2 = DF^2 + FC^2$; and as $DE = DF$ (*Ax.* 3), $EC = FC$; and $\therefore$ $AC + CB = AE + BF$, and $AC - CB = EC + CF$; and, therefore, AE or BF = half the sum of AC and CB, and EC or CF = half the difference of AC and CB.

EXERCISE 194.

If from the middle point of the arc of a circle which lies below the base of an inscribed triangle, perpendiculars be let fall on the sides, prove that the parts cut off on each side, measured from the vertex, are each equal to half the sum of the sides; and that the parts cut off, measured from the extremities of the base, are each equal to half the difference of the sides.

(Fig. 194, Plate XIII.)—Let ABC be a triangle inscribed in the circle $ACBD$; and let the perpendiculars DE and DF be let fall on AC and CB from D, the middle point of the arc ADB; then CE and CF are each equal to half the sum of AC and CB, and AE and BF are each equal to half the difference of AC and CB.

Because ACD is a triangle inscribed in the circle, and DE is a perpendicular on AC, the rectangle $CD \cdot DA$ = rectangle $DE \cdot DH$, DH being the diameter of the circle; and the rectangle $CD \cdot DB = CD \cdot DA$ = rectangle $DF \cdot DH$; and $\therefore DE = DF$. But $AD = DB$ (III. 29); and $\therefore$ (I. 47) $AE = BF$, and $EC = CF$; and $\therefore EC + CF = AC + CB$, and EC or CF = half the sum of AC and CB; and, as $AE = BF$, each is half the difference between AC and CB.

EXERCISE 195.

If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, and from the two points of intersection tangents be drawn to the circle, the squares on these tangents are together equal to the square on the straight line joining the points of intersection.

(Fig. 195, Plate XIII.)—Let $ABCD$ be a quadrilateral inscribed in a circle, and let the opposite sides on being produced intersect in the points M, R: $MR^2 = MS^2 + RE^2$, tangents drawn from M and R.

$MCR = DCB$ (I. 15); $\therefore MCR + BAD$ = two right angles (III. 22), and hence $MCR = AMR + ARM$ (I. 32). Draw CQ making $MCQ = ARM$, and meeting MR at Q. MCD = the angle at A, $\therefore DCQ + DMQ$ = two right angles, and $MDCQ$ can be circumscribed by a circle; for a similar reason $BCQR$ admits of being circumscribed by a circle; $\therefore MR \cdot RQ = DR \cdot RC = RE^2$ (III. 36), and $RM \cdot MQ = BM \cdot MC = MS^2$, but $MR \cdot RQ + RM \cdot MQ = MR^2$ (II. 2), $\therefore MR^2 = MS^2 + RE^2$.

EXERCISE 196.

If four straight lines intersect one another, and form four triangles, the circles which circumscribe them will all pass through the same point.

(Fig. 196, Plate XIV.)—Let the lines AB, AC, DB, and DE form the four triangles ABC, AEH, DBE, and DCH; let the circles $HCDP$ and $AEHP$, circumscribing the triangles DHC and AEH, intersect each other in P; the circles circumscribing the triangles ABC and DBE will also pass through P.

Join PA, PE, PH, PC, and PD.

Because the points P, H, C, and D are in the circumference of the circle $HCDP$, the angles PDC and PHC are together equal to two right angles (III. 22); and as PHC and PHA are also equal to two right angles (I. 13), $\therefore$ $PHA = PDC$ (*Ax.* 1); but $PHA = PEA$ (III. 21), $\therefore$ PDC or $PDB = PEA$, and the figure $PDBE$ is circumscribable by a circle, or the circle circumscribing EDB passes through P.

Again, because the angles PAE and PHE are together equal to two right angles, and PHE and PHD are together equal to two right angles, $PAE = PHD = PCD$ (III. 21); and $\therefore$ the quadrilateral $APCB$ is circumscribable by a circle; that is, the circle circumscribing ABC also passes through P. Therefore the four circles circumscribing the four triangles all pass through the point P.

EXERCISE 197.

ABC is a triangle, and P is the point of intersection of the perpendiculars from A, B, and C on the opposite sides of the triangle; then the circle which passes through the middle points of PA, PB, and PC will pass through the feet of the perpendiculars, and through the middle points of the sides of the triangle—that is, through the nine points.

(Fig. 197, Plate XIV.)—Let K, M, and N be the middle points of PC, PA, and PB; let E be the foot of the perpendicular from A on BC, and H the middle point of BC.

PBE is a right-angled triangle, and N the middle point of its hypotenuse BP; and $\therefore$ $NE = NP$; and $\therefore$ also the angle NEP = angle NPE. Again, because K is the central point of PC, the hypotenuse of the right-angled triangle CEP, $\therefore$

KE = *KP*, and the angle *KEP* = angle *KPE*; and ∴ *NEP* + *KEP* = *NPE* + *KPE*; that is, the angle *KEN* = angle *KPN* = angle *DPF* (I. 15). But *DPF* + *BAC* = two right angles, ∴ *KEN* + *BAC* = two right angles (*Ax.* 1). Again, since *AP* and *PB* are bisected in *M* and *N*, *MN* is parallel to *AB*, and ∴ (I. 29) the angle *PMN* = angle *PAB*. For a similar reason the angle *PMK* = angle *PAC*, and (*Ax.* 1) the whole angle *NMK* = *BAC*; and ∴ *ENMK* is circumscribable by a circle (III. 22, *Cor.*) Again, since *PC*, *BC*, and *PB* are bisected in *K*, *H*, and *N*, the figure *NPKH* is a parallelogram, and the angle *NPK* = *NHK*; and ∴ the angle *NEK* = *NHK*, and hence *E* and *H* are in the same segment; and ∴ a circle will pass through *M*, *N*, *E*, *H*, and *K*. In like manner it can be shown that it will pass through *D*, *F*, and the middle points of *AB* and *AC*.

EXERCISE 198.

The three straight lines joining the feet of the three perpendiculars from the angles of a triangle on the opposite sides make equal angles with the sides; and the rectangle under the segments of any side is equal to the rectangle of the lines which make equal angles with it at the foot of the perpendicular.

(Fig. 198, Plate XIV.)—Let *ABC* be a triangle; *AF*, *BD*, *CE* perpendiculars from *A*, *B*, *C* on the sides *BC*, *AC*, *AB*; and *FD*, *FE*, *EF* the three straight lines joining the feet of the perpendiculars; then, *firstly*, the angle *DFC* = angle *BFE*, the angle *BEF* = angle *DEA*, and the angle *ADE* = angle *CDF*, and the complements of these equal angles are equal—namely, the angle *DFH* = *EFH*, the angle *FEH* = *DEH*, and the angle *EDH* = angle *FDH*; and, *secondly*, the rectangle $BF \cdot FC = EF \cdot FD$, the rectangle $AE \cdot EB = DE \cdot EF$, and the rectangle $AD \cdot DC = ED \cdot DF$.

Firstly—Because the angles *HDC* and *HFC* are right angles (*Hyp.*), the quadrilateral *FHDC* is circumscribable by a circle (III. 22, *Cor.*); and ∴ the angle *DFC* = angle *DHC* (III. 21); but *DHC* = *EHB* (I. 15), ∴ *DFC* = *EHB*. Again, because the angles *BEH* and *BFH* are right angles, the figure *EHFB* is circumscribable by a circle, and ∴ the angle *EHB* = angle *EFB*; but *EHB* has been shown to be equal to DFC, $\therefore$ $EFB = DFC$; and, as the angles AFB and AFC are right angles, and $BFE = DFC$, the angles AFE and AFD are also equal (*Ax.* 3). In the same way it can be shown that the angle FDC = angle EDA, the angle EDH = angle FDH, the angle $AED = BEF$, and the angle FEH = angle DEH.

SECONDLY—From BF, supposed to be the greater, cut off FG equal to FC; and from EF, supposed to be the greater, cut off FK equal to FD; and join GK. Now, GF and FK and the angle GFK are respectively equal to CF and FD and the angle CFD; and $\therefore$ (I. 4) the triangles GFK and CFD are equal in every respect, and the angle FGK = angle FCD. Again, because the angles AEC and AFC are right angles (*Hyp.*), $AEFC$ is circumscribable by a circle, and the angles AEF and ACF = two right angles; but AEF and BEF = two right angles; $\therefore$ the angle BEF = angle ACF; and as $FGK = ACF$, $\therefore$ $BEF = FGK$, and the figure $BEKG$ is circumscribable by a circle, and, therefore, the rectangle $BF \cdot FG = EF \cdot FK$; that is, the rectangle $BF \cdot FC$ = rectangle $EF \cdot FD$. In like manner it can be shown that $AE \cdot EB = DE \cdot EF$; and $AD \cdot DC = ED \cdot DF$; and $\therefore$ (*Ax.* 1) $AE \cdot EB + AD \cdot DC + BF \cdot FC = DE \cdot EF + DE \cdot DF + EF \cdot DF$.

Note.—Of all the triangles that can be inscribed in the triangle ABC, EDF has the least perimeter, since its sides make equal angles with AB, BC, AC (*Ex.* 149, PART I.).

EXERCISE 199.

If any two chords be drawn through the middle point of a given chord of circle, the straight lines joining their extremities cut off equal parts from the given chord.

(Fig. 199, Plate XIV.)—Let FQM, RQL be the chords drawn through Q, the middle point of any other chord AQK, and FHL, RXM the lines joining their extremities and cutting AQK in H, X: then $HQ = XQ$.

Draw RS parallel to AK; join SF, SH, SQ, and draw QW perpendicular to SR.

QW passes through the centre and bisects SR in W. Triangles SWQ, RWQ are equal (I. 4), $\therefore$ the angles QSW, QRW are equal, but since SR is parallel to AK, $QSW = SQH$, $\therefore$ $QRW = SQH$. Again, because $FSRL$ is a quadrilateral inscribed in a circle the opposite angles at F and R equal two right angles, but SQH equals the angles of the quadrilateral at R, $\therefore$ $SFHQ$ is inscribable in a circle, and $QSH = QFH = QRX$, but $SQH = QRW = RQX$, and $QS = QR$, $\therefore$ the triangles QSH, QRX are equal (I. 26), and hence $HQ = XQ$.

EXERCISE 200.

If a point be taken in the circumference of a circle in which a quadrilateral figure is inscribed, and if from the point perpendiculars be let fall on the sides, prove that the rectangle under the perpendiculars on one pair of opposite sides is equal to the rectangle under the perpendiculars on the other pair of opposite sides.

(Fig. 200, Plate XIV.)—Let $ABCD$ be the quadrilateral, and QL, QF, QS, QR the perpendiculars on the sides AD, BC, AB, CD respectively from a point Q, in the circumference of the circumscribing circle: then $LQ \cdot QK = QR \cdot QS$.

Join SK, BQ, RL, DQ, and on QL, QR take $QF = QK$, and $QZ = QS$ respectively; join FZ.

In the quadrilaterals $QKBS$, $QLDR$, because the whole angles at S, K, L, R, are right, and $SBC = CDA$, they are inscribable in circles, and equiangular; $\therefore$ $SQK = RQL$, and the triangles SQK, ZQF are equal in all respects (I. 4). Now, $QDC = QBC$ (III. 21), but $QDC = QLR$, and $QBC = QSK$, $\therefore$ the triangles SQK and QRL are equiangular, or ZQF, QRL are equiangular, but since $QZF = QSK = QLR$ the quadrilateral $RLFZ$ is inscribable in a circle, and $\therefore$ $LQ \cdot QF = RQ \cdot QZ$; that is, $LQ \cdot QK = QR \cdot QS$.

EXERCISE 201.

If perpendiculars be drawn from any point on the circumference of a circle to two tangents and their chord of contact, the square on the perpendicular to the chord is equal to the rectangle under the other two perpendiculars.

(Fig. 201, Plate XIV.)—Let BA, BC, be tangents to the circle ADC; D any point in the circumference from which the perpendiculars DK, DN, DM are drawn to AC, AB, BC respectively: $DK^2 = DM \cdot DN$.

Join DA, DC, and on DM take $DS = DN$; join SK.

In the quadrilaterals $ANDK$, $CMDK$, because the whole angles at N, K, M are right, and the whole angles A, C are equal, the angles NDK, MDK are equal, and the quadrilaterals are inscribable in circles. Again, $NAD = KCD$ (III. 32), but since $ANDK$, $CMDK$ are inscribable in circles, $NKD = NAD = KCD = DMK$, but $NKD = DKS$ (I. 4), $\therefore DKS = DMK$, and hence DK is a tangent to the circle passing through K, S, M, $\therefore DK^2 = MD \cdot DS = MD \cdot DN$ (III. 36).

EXERCISE 202.

If two circles touch externally and a common tangent be drawn not passing through the point of contact: then the square of this tangent shall be equal to the rectangle of the diameters of the circles.

(Fig. 202, Plate XIV.)—Let the two circles FNR, FMD, which touch in F, have a common tangent MN; let FR, FD be their respective diameters, and S, H their centres respectively. From F draw the common tangent FL meeting MN in L, and join LS, LH.

Now, $LM = LF = LN$, and LS, LH bisect the angles FLN, FLM respectively, $\therefore HLS$ is a right angle, as also is HFL, $\therefore$ in the right-angled triangle HLS, $HF \cdot FS = LF^2$, or $4HF \cdot FS = 4LF^2$, but $4HF \cdot FS = FD \cdot FR$, and $4LF^2 = MN^2$; $\therefore MN^2 = FD \cdot DR$.

Otherwise—Let QW be the common tangent and let S be the centre of the larger circle, the point Q being in its circumference; join SQ, HW, and draw HK parallel to WQ, meeting SQ in K.

WK is evidently a rectangle, $\therefore WQ^2 = HK^2 = (SH^2 - SK^2) = (SH + SK) \cdot (SH - SK)$, but $SH + SK = FR$, and $SH - SK = FD$, $\therefore WQ^2 = FR \cdot FD$.

EXERCISE 203.

In equiangular triangles the rectangles under the non-corresponding sides about equal angles are equal. (See Euclid, VI. 4).

(Fig. 203, Plate XIV.)—Place the triangles NKS, LKM so that the equal angles at K shall be vertically opposite, and the non-corresponding sides NK and MK, SK and LK shall be in the same straight line. Now, the angles at N, L are opposite the corresponding sides SK, MK, and are $\therefore$ equal, hence the points S, N, L, M are in the circumference of a circle, $\therefore SK \cdot KL = NK \cdot KM$ (III. 35).

In the same way it may be proved of the remaining non-corresponding sides.

EXERCISE 204.

The exterior angle CBD of the triangle ABC is bisected by the line BE, which cuts the base produced in E: show that the square on BE, together with the rectangle AB, BC, is equal to the rectangle AE, EC.

(Fig. 204, Plate XIV.)—Make $BD = BC$ and join DE; draw AM meeting EB produced in M and making angle MAB = angle CEB.

Triangles BCE, BDE are equal (I. 4), and because the angle $ABM = DBE = CBE$, and $MAB = CEB$, $AMB = ECB$; ∴ $AMBC$ is circumscribable by a circle; and since $DEB = MAB$, the points A, M, D, E are in the circumference of a circle, ∴ $AB \cdot BD = MB \cdot BE$ (III. 35); that is, $AB \cdot BC = MB \cdot BE$; to each add BE^2, and $AB \cdot BC + BE^2 = MB \cdot BE + BE^2 = ME \cdot EB = AE \cdot EC$ (III. 36).

EXERCISE 205.

Through a fixed point A on a circle, a chord AB is drawn, and produced to a point M, so that $AB \cdot AM$ is constant: find the locus of M.

(Fig. 205, Plate XIV.)—From M draw a line making an angle with AM equal to that in the segment on the same side of AM, as the line is drawn: this line is the locus of M.

For, take any point S, and draw AS cutting the circle in R; join BR.

Now, because angles ARB, AMS are equal, $SRBM$ is capable of being inscribed in a circle, ∴ $SA \cdot AR = AB \cdot AM$ (III. 36).

The same may be proved if the point be taken in SM produced, and on the other side of AM.

EXERCISE 206.

If three circles intersect one another, their three chords of intersection pass through the same point.

(Fig. 206, Plate XIV.)—Let A, B, O be the centres of three circles which intersect one another. Let the chords CD, EF intersect in P; join HP: then HP, if produced, will pass through G.

If possible, let it not pass through G, but through some other point, K.

Because the chords CD, EF, of the circle whose centre is A, intersect in P, $CP \cdot PD = EP \cdot PF$ (III. 35). For the same reason $HP \cdot PK = CP \cdot PD$, both CD and HPM being chords of the circle whose centre is O; and $EP \cdot PF = HP \cdot PM$, both EF and HPM being chords of the circle whose centre is B. But, since $CP \cdot PD = EP \cdot PF$, $HP \cdot PK = HP \cdot PM$; and $\therefore PK = PM$, which is absurd. Hence HP, if produced, must pass through G; that is, the three common chords of the three intersecting circles pass through the same point P.

EXERCISE 207.

If any arc of a circle be divided into two parts, the sum of their chords is a maximum when the arc is bisected.

(Fig. 207, Plate XIV.)—Let C be the point of bisection of the arc ACB, and let CA, CB be joined: then the sum of CA, CB is greater than that of any other chords DA, DB drawn from any point D in the arc.

With C as centre, and CA or CB as radius, describe a circle ABF, AC produced meeting the circumference at F, and AD produced meeting it at E; join EB.

Angle $ADB = ACB =$ twice AEB (III. 21 and 20), $\therefore DBE = DEB$ (I. 32), hence $DB = DE$, $\therefore AF = AC + CB$ and $AE = AD + DB$, but AF is greater than AE (III. 15), $\therefore CA + CB$ is greater than $AD + DB$.

EXERCISE 208.

Prove that a circle which passes through the foot of the perpendicular on the base from the vertical angle, and the feet of those which are let fall from the extremities of the base on the bisector of the vertical angle, will also pass through the middle point of the base.

(Fig. 208, Plate XIV.)—Let BD, CF, AG be the perpendiculars on the base AC, and the bisector of the vertical angle of triangle ABC, respectively, and let E be the middle point of the base: then G, D, F, E are in the circumference of the same circle.

Join DG, EF, EG, and produce CF to H, meeting AB in that point.

Triangles HFB, CFB are equal (I. 26), $\therefore HF = FC$, but $AE = EC$, hence EF, which joins the middle points of CH, CA, is parallel to AH, $\therefore$ angle $EFG = ABG$; but since AGB, *ADB* are right angles (*Cons.*), the points *A*, *B*, *D*, *G* are in the same circumference, and hence *ADG* = *ABG*, ∴ *EFG* = *EDG*, ∴ the circle which passes through *G*, *D*, *F* also passes through *E*, the middle point of the base.

EXERCISE 209.

Given a chord of a circle and two points on the circumference underneath it, find a third point on the circumference above it such that if it be joined to the two given points the part of the chord intercepted shall be of given length.

(Fig. 209, Plate XIV.)—Let *MN* be the chord, and *A*, *C* the points beneath it; let *CH*, drawn parallel to *MN*, be the given length.

Join *AC*, *AH*, and on *AH* describe a segment of a circle containing an angle equal to that in the segment *AMC*, cutting *MN* in *W*; join *AW* and produce it to cut the circumference in *B* : *B* is the point required.

Join *WH*, *BC*, the latter cutting *MN* in *Q*.

Angles *AWH*, *ABC* are equal, ∴ *WH* is parallel to *BC*; but *WQ* is parallel to *CH* (*Cons.*), ∴ *WHCQ* is a parallelogram, hence *WQ* = *CH*.

EXERCISE 210.

Through a point given in position within the legs of an angle draw a straight line which shall cut off on the legs portions measured from the vertex so that the rectangle under them shall be given.

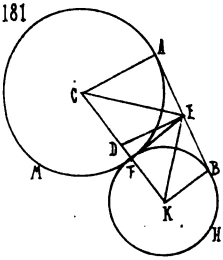
(Fig. 210, Plate XIV.)—Let *M* be the given point within the angle *HNW*.

Join *MN*, and draw *NS* making angle *SNH* = *MNW*, so that *SN·NM* shall be equal to the given rectangle; join *SM*, and on it describe a segment of a circle containing an angle equal to *SNH*; let this segment cut *NW* in *W*; join *WM* and produce it to meet *NH* in *H* : *HMN* is the line required.

HNS = *HWS*, ∴ the points *S*, *H*, *N*, *W* are in the circumference of a circle, and since angle *SNH* = *MNW* (*Ex.* 186), *NH·NW* equals *SN·NM* equals the given rectangle.

PLATE XIII

181



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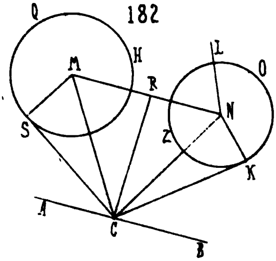

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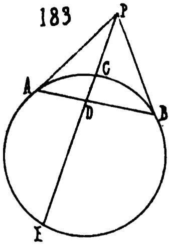

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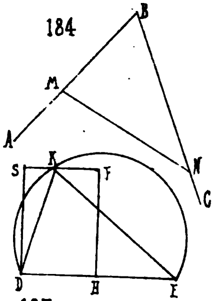

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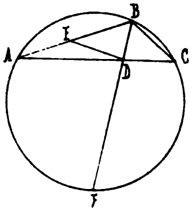

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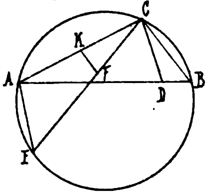

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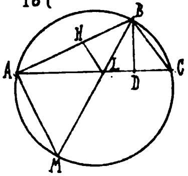

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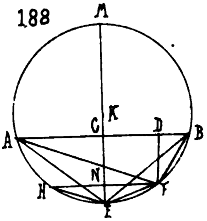

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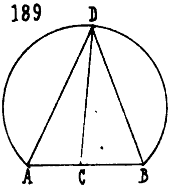

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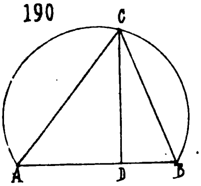

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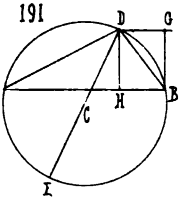

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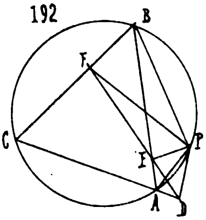

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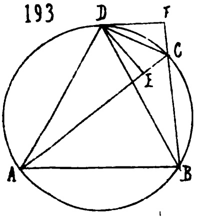

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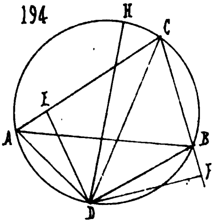

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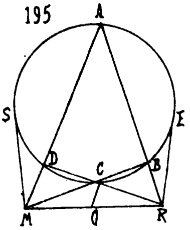
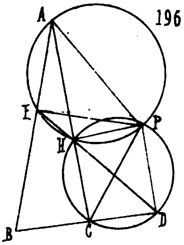


PLATE XIV.

196



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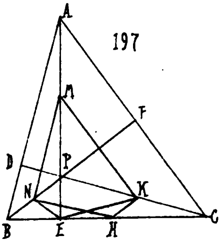

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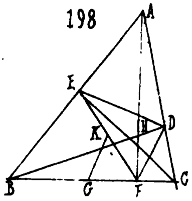

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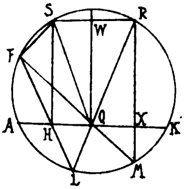

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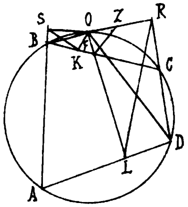

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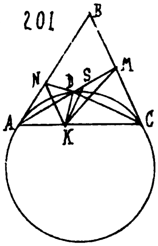

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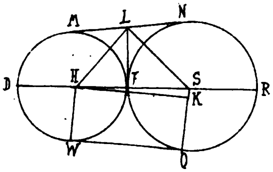

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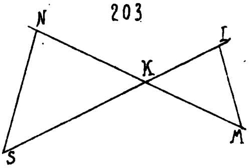

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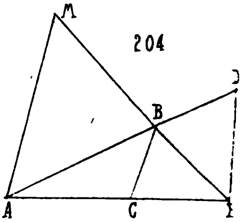

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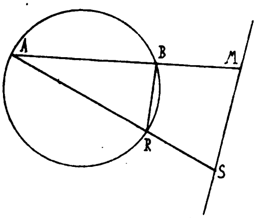

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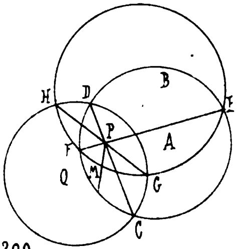

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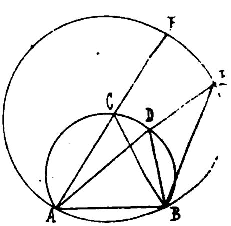

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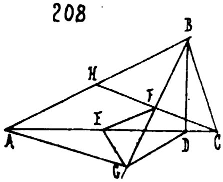

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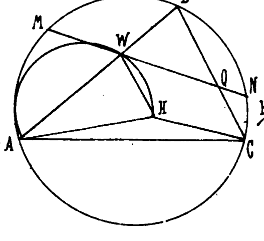

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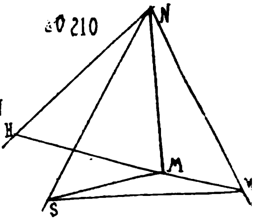

Fourth Book.

PLATE XIV.

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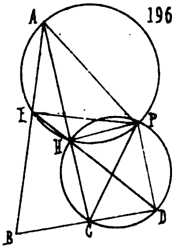

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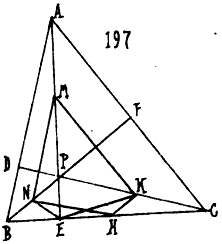

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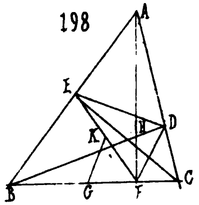

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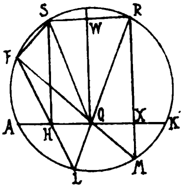

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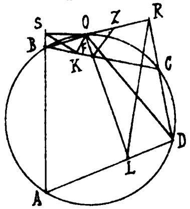

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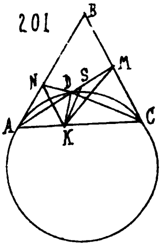

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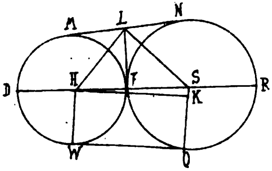

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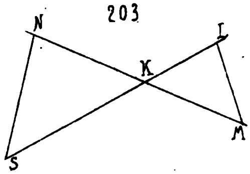

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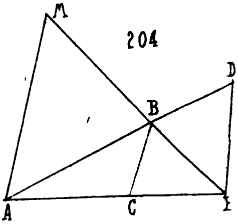

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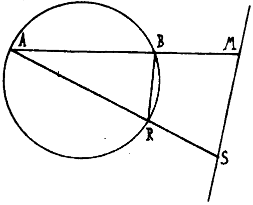

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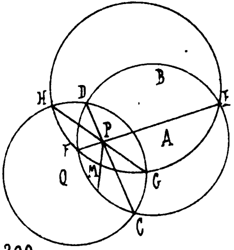

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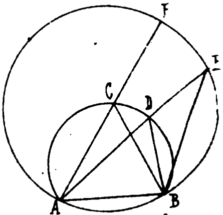

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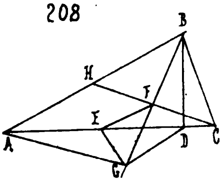

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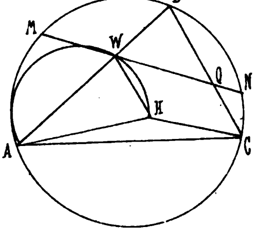

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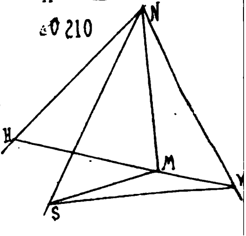


Fourth Book.

EXERCISES UPON EUCLID.

FOURTH BOOK.

EXERCISE 211.

Inscribe a square in a given quadrant.

(Fig. 211, Plate XV.)—Let A be the centre of the circle of which AKH is the given quadrant.

Draw AL bisecting the angle at A and meeting the arc in L; draw LW, LM perpendiculars to AK, AH respectively: $AWLM$ is the square required.

WAL is half a right angle (*Cons.*), and the angle at W is right, $\therefore$ WLA is half a right angle (I. 32), and hence $WL = WA$ (I. 6), $\therefore$ $AWLM$ is a square (I. 34), and is inscribed in the given quadrant.

EXERCISE 212.

Inscribe a square in the space included between two equal circles which intersect.

(Fig. 212, Plate XV.)—Join M, N, the centres of the circles, and from S, the point of bisection of the line MN, draw SB meeting one of the circumferences at B, and draw the chord BC at right angles to MN, cutting it at W; make $SR = SW$, and through R draw the chord AK at right angles to MN; join AB, KC: $ABCK$ is the square required.

$NR = MW$, $\therefore$ AK equals BC (III. 14), and is parallel to it, $\therefore$ $ABCK$ is a parallelogram. Now, since $AR = BW$, and the angles at R, W are right, and BSN equals half a right angle, $\therefore$ the angles of the figure at A, B are right, $AB = RW = BC$, and hence $ABCK$ is a square.

EXERCISE 219.

Describe a circle which shall pass through one angle and touch two sides of a given square.

(Fig. 219, Plate XV.)—Let $ABCD$ be the given square. Draw AC, one of the diagonals; bisect the angle ACD by the line CE; make the angle CEK equal to the angle ECK; then K is the centre of the circle. From K draw KF perpendicular to AB.

Because the angle KEC = angle KCE (*Cons.*), $KC = KE$ (I. 6); and because the angle AKE = the two angles KCE and KEC, it is equal to the angle EAK = half a right angle, and ∴ $KE = AE$; and (I. 32) the angle AEK is a right angle; and as the angle AEK = angle AFK, and angle KAF = angle KAE, and AK common, the perpendicular KE = perpendicular KF; and as $KC = KE = KF$, K is the centre of the circle passing through C and touching AD and AB respectively in E and F (III. 9 and 16).

EXERCISE 220.

If tangents be drawn through the extremities of two diameters of a circle, the parallelogram thus formed will be a rhombus.

(Fig. 220, Plate XV.)—Let AB and DE be two diameters of the circle $ADBE$, and let tangents be drawn through the points A, D, B, E; then the figure $FKHG$, formed by these tangents, is a rhombus.

Since $KHGF$ is a parallelogram, $KH = FG$, and $KF = HG$ (I. 34); and since $KHGF$ is a quadrilateral described about the circle, $KH + FG = KF + HG$ (*Ex.* 6); and ∴ (*Ax.* 7) $KH = KF = FG = HG$; and the figure $KHGF$ is a rhombus.

Note.—If AB and DE be at right angles, the figure $KHGF$ will evidently be a square (IV. 7); and it is equally evident that a square is the only rectangle that can be described about a circle.

EXERCISE 221.

The centres of the inscribed and circumscribed circles of an equilateral triangle coincide, and the diameter of one is double that of the other.

(Fig. 221, Plate XV.)—Let ABC be an equilateral triangle, having one circle inscribed and another circumscribed, From A, B, C let fall the perpendieulars AD, BE, CF. These perpendiculars bisect the sides and also the angles (*Ex.* 10, PART I.); and each perpendicular is so divided that the longer segment is double the shorter (*Ex.* 163, PART I.); that is, $AK = 2KD$.

The three perpendiculars intersect in K; and since the angles at A, B, C are bisected, K is the centre of the inscribed circle (IV. 4). Again, since the sides are bisected perpendicularly by EK, FK, DK, the point K is the centre of the cricumscribed circle (IV. 5). But it has been shown that $AK = 2KD$, and AK is the radius of the circumscribed circle, and KD that of the inscribed circle (IV. 4 and 5), and hence the diameter of one is double that of the other.

EXERCISE 222.

ACDB is a semicircle, AB being the diameter, and the two chords AD and BC intersect at E: show that if a circle be described round CDE it will cut the semicircle ACDB, or the circle of which it is the half, at right angles.

FIRST METHOD.

(Fig. 222, Plate XV.)—Draw CF a tangent to the circle circumscribing CDE, and meeting AB in F, and join C and the centre K of the circle $CEDG$, and also join C and D.

Because CF is a tangent and CB a chord, the angle BCF = angle CDE (III. 32); and the angle CDE = angle CBA (III. 21); and $\therefore$ the angle BCF = angle CBA (*Ax.* 1), and $FC = FB$ (I. 6). In the same manner it may be shown that $FD = FA$; but $FC = FD$ (III. 36), $\therefore$ F is the middle point of AB, and hence the circle $CEDG$ cuts the semicircle $ACDB$ at right angles.

Note.—One circle is said to cut another at right angles when the radius of one is a tangent to the other: in other words, the circles are said to cut each other orthogonally.

SECOND METHOD.

Join the middle point F of AB with the point C; then FC is a tangent to the circle $CEDG$. Now, because $FC = FB$, the angle FCB = angle FBC (I. 5); but the angle FBC = ADC (III. 21); and $\therefore$ angle ADC = angle FCB (*Ax.* 1), and $\therefore$ FC is a tangent to the circle $CEDG$ (III. 32, *Cor.*). In the same way it can be shown that FD if joined will be a tangent. Hence the circles cut each at right angles or orthogonally.

EXERCISE 223.

C is the centre of a circle; CA, CB are two radii at right angles; from B any chord BP is drawn cutting CA at N; a circle is described round ANP: show that it will be touched by BA.

(Fig. 223, Plate XV.)—The angle at P equals half a right angle (III. 20), and ∴ equals BAC, and hence the circle circumscribing ANP touches BA at A (III. 32).

EXERCISE 224.

The common chord ML of two circles is produced to P; PA touches one of the circles at A, PBC is any secant of the other: show that the circle which passes through A, B, and C touches the circle to which PA is a tangent.

(Fig. 224, Plate XV.)—Let the circle be described passing through ABC.

$MP{\cdot}PL = CP{\cdot}PB$, ∴ $CP{\cdot}PB = PA^2$, and hence PA is a common tangent to the circles MAL, BAC, ∴ they touch at the point A.

EXERCISE 225.

If the chord of a quadrant be made the diameter of a semicircle, and from its extremities two straight lines be drawn to any point in the circumference of the semicircle; the segment of the greater line intercepted between the two circumferences shall be equal to the less of the two lines.

(Fig. 225, Plate XV.)—Let H be the centre of the quadrant NKM; join NM, and on it describe the semicircle NWM, and from any point W in this, draw WN, WM, the latter cutting the arc of the quadrant in K: then $WK = WN$.

Join NK.

Because NKM is the angle in a quadrant it is equal to a right angle and a-half, being equal to half the angle at the centre, ∴ NKW equals half a right angle, but NWK is a right angle (III. 31), ∴ KNW is also half a right angle (I. 32), and hence $WK = WN$.

EXERCISE 226.

If two circles cut each other, and any two points be taken in the circumference of one of them through which lines are drawn from the *points of intersection and produced to the circumference of the other: the straight lines joining the extremities of those which are drawn through the same point are equal.*

(**Fig. 226, Plate XVI.**)—Let the circles *HBK*, *MHK*, cut each other in *H*, *K*, and in *HBK* take any two points *N*, *B*, through which draw *HNS*, *HBW*, *KNM*, *KBL*; join *MS*, *WL*: then *MS* = *WL*.

BKN = *BHN*, ∴ the arcs *ML*, *WS* are equal; to each add the arc *LS*, then the arcs *MS*, *WL* are equal, ∴ *MS* = *WL* (III. 29).

EXERCISE 227.

If from the extremities of any chord of a circle perpendiculars to it be drawn and produced to cut a diameter; and from the points of intersection with the diameter lines be drawn to a point in the chord, so as to make equal angles with it: these lines together will be equal to the diameter.

(**Fig. 227, Plate XVI.**)—Let *DC* be any chord of the circle *HDB*; draw *DK*, *CW* perpendicular to it meeting the diameter *AB* in *K*, *W*; from which let the lines *KS*, *WS* be drawn making equal angles with *DC*: then *KS* + *WS* = the diameter.

Draw *CEH* through the centre *E*, meeting the circumference at *H*; *DK* produced will also meet it at that point, because the angle at *D* is right; produce *KS* to meet *WC* produced at *M*.

Triangles *HEK*, *CEW* are equal (I. 26), as also are triangles *CSW*, *CSM*, ∴ *HK* = *CW* = *CM*, but *HK* is parallel to *CM*, and hence *KM* is equal and parallel to *HC* (I. 33)—that is, *KS* + *WS* = the diameter *HC*.

EXERCISE 228.

Describe a circle which shall pass through two given points A, B, and cut off from a given straight line HS a chord equal to a given line DK.

(**Fig. 228, Plate XVI.**)—Join *AB* and produce it to meet *HS* at *S*; produce *DK* to *N* making $DN \cdot NK = AS \cdot SB$; on *HS* take *SW* = *DN*, making *WM* = *DK*; join *AW*, *BM*.

Now, since $AS \cdot SB = DN \cdot NK = WS \cdot SM$, the quadrilateral *ABMW* is circumscribable by a circle, which is the circle required, for it cuts off from *HS* the part *WM* = *DK*.

EXERCISE 229.

Describe a circle which shall pass through two given points, so that the tangent drawn to it from another given point may be of given length.

(**Fig. 229, Plate XVI.**)—Let A, B be the points through which the circle must pass, and L the point from which the given tangent LD is to be drawn.

Join AL and cut it in K making $AL \cdot LK = LD^2$; join BA, BK, and circumscribe AKB by a circle (IV. 5): then this is the circle required.

Draw the tangent LD.

The proof is evident from the construction.

EXERCISE 230.

Describe a circle which shall have its centre in a given straight line, and cut off from two given straight lines chords of equal given length.

(**Fig. 230, Plate XVI.**)—Let the given lines MC, KC, not being parallel, meet at C; and CS, which bisects the angle at C, meet the line in which the centre is to be, at S; from S draw SW perpendicular to CK, and in CK take RH equal to the given line, so that it may be bisected at W; join SH, and with S as centre, and SH as radius, describe a circle cutting CM in V, Q : QVH is the circle required.

This circle also passes through R, because $SH = SR$.

Now, because CQ, CR are drawn making equal angles with CS, which is drawn to the centre, $\therefore CQ = CR$ (III. 8); for a similar reason $CV = CH$, $\therefore QV = RH$, and hence QVH is the circle required.

EXERCISE 231.

If O be the centre of the circle inscribed in the triangle ABC, and AO be produced to meet the circumscribed circle at F, show that FB, FO, and FC are all equal.

(**Fig. 231, Plate XVI.**)—Because O is the centre of the inscribed circle, AF bisects the angle BAC (IV. 4), $\therefore FB = FC$ (III. 26 and 29). Again, because $FBC = FAC = BAO$, and $OBC = ABO$, $FBO = BAO + ABO = FOB$, $\therefore FO = FB$ and hence $FB = FO = FC$.

EXERCISE 232.

If a circle be inscribed in a right-angled triangle, the excess of the sides containing the right angle above the hypotenuse is equal to the diameter of the inscribed circle.

(Fig. 232, Plate XVI.)—Let D be the centre of the circle inscribed in the triangle ABC right-angled at B; let it touch the hypotenuse at K, and BA, BC at N, M respectively: then $BA + BC$ less the diameter of the circle, equals AC.

For, join DN, DM.

Then, since the angles at N, B, M are right, and $DN = DM$, NM is a square; $\therefore$ $BN + BM = DN + DM =$ the diameter of the circle. Now, because $AN = AK$, and $CM = CK$, and $BN + BM$ equals the diameter, it is manifest that the excess of $BA + BC$ above AC equals the diameter of the circle.

EXERCISE 233.

Find the locus of the centres of the circles which are inscribed in all right-angled triangles on the same hypotenuse.

(Fig. 233, Plate XVI.)—Let ABC be a right-angled triangle on the hypotenuse AC, and D the centre of the inscribed circle.

Join AD, CD.

These lines bisect the angles from which they are drawn (**IV. 4**), $\therefore$ ADC equals a right angle and a-half, which is the angle in a quadrant, and hence the locus required is the arc of a quadrant described on the hypotenuse AC.

EXERCISE 234.

If the three perpendiculars of a triangle ABC intersect in H, and meet the circumscribed circle in E, F, D: show that the lines HF, HE, HD are bisected by the sides of the triangle.

(Fig. 234, Plate XVI.)—Let HF cut AC in M and AE cut BC in N; join AF.

Triangles CBM, CAN are equiangular, having a right angle and a common angle, $\therefore$ $CAH = CBF = CAF$, and hence triangles MHA, MFA are equal (**I. 26**), $\therefore$ $MH = MF$.

It may be similarly shown that HE, HD are bisected at their points of intersection with the remaining sides.

EXERCISE 235.

A circle is described round triangle ABC, and from any point P of its circumference perpendiculars are drawn to BC, CA, AB, which meet the circle again at D, E, F: show that triangles ABC, DEF are equal in all respects.

(**Fig. 235, Plate XVI.**)—Let the perpendiculars meet the sides *BC*, *AC*, *AB* at *H*, *L*, *M* respectively; join *AP*.

Then *PLCH* is circumscribable by a circle; and *BAC* = *EPF* (I. 32) = *EDF* (III. 21), and *BCA* = *EPD* = *EFD*, hence triangles *ABC*, *DEF* are equal in all respects (I. 26 and III. 29).

EXERCISE 236.

The perpendicular from the vertex on the base of an equilateral triangle is equal to the side of an equilateral triangle inscribed in a circle whose diameter is the base.

(**Fig. 236, Plate XVI.**)—With the base *AC*, of the equilateral triangle *ABC*, as diameter, describe the circle *ADC*, cutting *BC* in *D*; join *AD*.

Because the angle at *D* is right, and *ACD* the angle of an equilateral triangle, *AD* is a perpendicular on the base *CB*, and is the side of an equilateral triangle inscribed in *ADC*.

EXERCISE 237.

The perpendiculars from the angles A and B of a triangle on the opposite sides meet at D; the circles described round ADC and DBC cut AB or AB produced at the points E and F: show that AE is equal to BF.

(**Fig. 237, Plate XVI.**)—Let the perpendiculars on *CA*, *CB* meet these sides at *M*, *N* respectively, and let *CD* produced meet *BA* at *H*; join *CE*, *CF*.

Because the angles at *N*, *H* are right angles, and the angle at *B* is common to the triangles *BNA*, *BHC*, *BAN* = *BCH* (I. 32), and since *DAEC* is a quadrilateral inscribed in a circle *BAN* = *ECH*, ∴ *BCH* = *ECH*, ∴ triangles *BCH*, *ECH* are equal (I. 26), and *BH* = *EH*. Again, *HCF* = *HBM* = *HCA*, ∴ triangles *HCF*, *HCA* are equal (I. 26), and *HF* = *HA*, but *BH* = *EH*, ∴ *AE* = *BF*.

EXERCISE 238.

If the base and vertical angle of a triangle be given, prove that the line joining the feet of the perpendiculars from the base angles on the opposite sides is also given.

(Fig. 238, Plate XVI.)—Let the base AB and vertical angle of the triangle ACB be given; let AD, BE be perpendiculars from A and B on the opposite sides, BC and AC; and let DE be the line joining the feet of these perpendiculars; then DE is also given.

On AB describe the semicircle $AEDB$. The angle at C is constant (III. 21); the angle ADC is (*Cons.*) a right angle; and ∴ the angle EAD or EBD is (I. 32) constant, since it is the complement of the vertical angle at C; and ∴ the arc ED and the chord DE are also constant (III. 26 and 29), and therefore DE is given.

EXERCISE 239.

Find the centre of the circle which touches the two circles described on the sides about the right angle in a right-angled triangle.

(Fig. 239, Plate XVI.)—Let AMB, CNB be the circles described on the sides AB, CB of the triangle ABC right-angled at B.

Bisect AC in D: D is the centre required. For, draw DF, DE perpendiculars to AB, CB respectively, and produce them to meet the circumferences at M, N.

Now, because DF is drawn parallel to CB, from the middle point of AC, it bisects AB, and is equal to half CB; for a similar reason DE bisects CB and equals half AB (*Ex.* 124, Part 1); hence F, E are the centres of the circles AMB, CNB respectively, ∴ $FM = FB$, and $EN = EB$, but DB is a rectangle; ∴ $DF + FB = DE + EB$—that is, $DM = DN$, and hence D is the centre required.

EXERCISE 240.

Given the angles of a triangle and the radius of the inscribed circle: to construct the triangle.

(Fig. 240, Plate XVI.)—Let the angles A, B, C be given; and let KD be equal to the radius of the inscribed circle; it is required to construct the triangle.

At K, with the line KD, make the angle DKA equal to the complement of half the angle A (I. 23), and the angle DKC equal to the complement of half the angle C; and through D draw AC at right angles to KD. Make the angle KAB equal to KAD, and the angle KCB equal to KCD; then ABC is the triangle required.

From K draw KE and KF perpendicular to BC and AB respectively. It is evident from I. 26 or I. 47 that $KD = KE = KF$, and from I. 32 that the angles at A, B, C are equal to the given angles; and $\therefore$ ABC is the required triangle.

Note.—The proof of this proposition is evident from the construction and IV. 4.

EXERCISE 241.

Given the vertical angle, and the radii of the inscribed and circumscribing circles: to construct the triangle.

(Fig. 241, Plate XVII.)—With the radius of the circumscribing circle, describe a circle, and let AC be a chord subtending the given vertical angle; on AC describe a segment of a circle containing an angle equal to the given angle, together with half its supplement; draw AH perpendicular to AC, and equal to the radius of the inscribed circle; draw HM parallel to AC meeting the arc of the segment at M, and join MC; draw CB meeting the circumference of the circle at B, and making $BCM = MCA$; join AB: ABC is the triangle required.

Draw MN, MQ, MR perpendiculars to AC, AB, BC respectively.

$BAM + BCM$ equals half the supplement of the given angle at B, $\therefore BAM + BCM = MAC + MCA$; but $BCM = MCA$ (*Cons.*), $\therefore BAM = MAC$, and since $MN = MQ = MR$ (I. 26), $\therefore ABC$ is the required triangle.

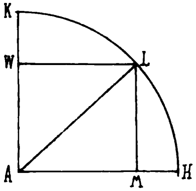
EXERCISE 242.

Given the vertical angle, the line drawn to the base bisecting that angle, and the difference between the base and the sum of the sides: to construct the triangle.

(Fig 242, Plate XVII.)—Let DAE be the given angle, AH the line bisecting it; make AF and AG each equal to half

PLATE XV.

211



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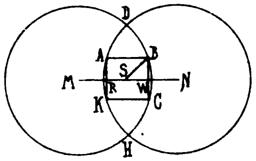

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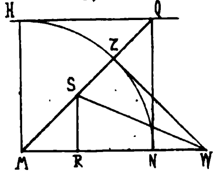

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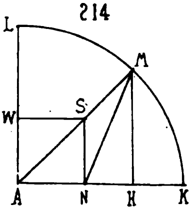

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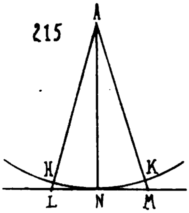

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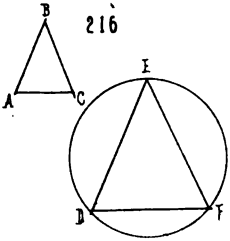

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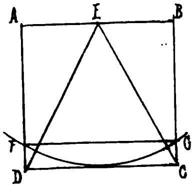

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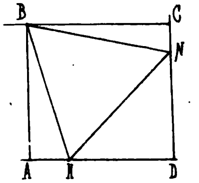

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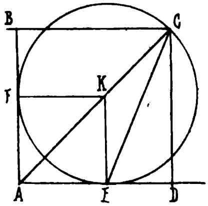

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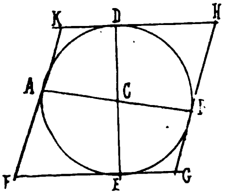

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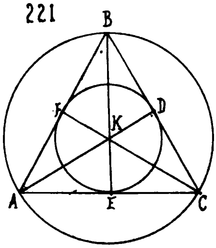

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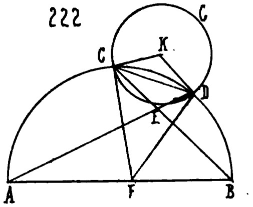

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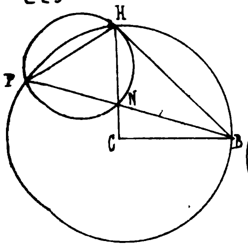

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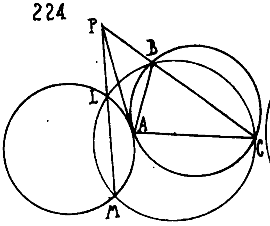

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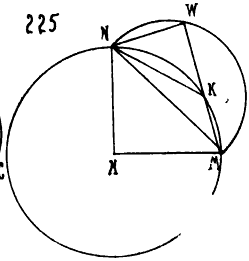

PLATE XVI.

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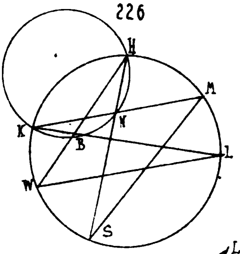

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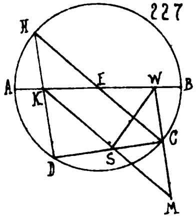

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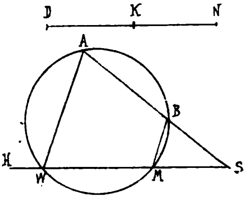

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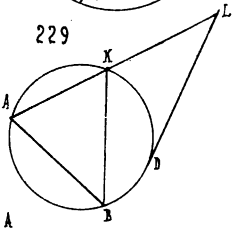

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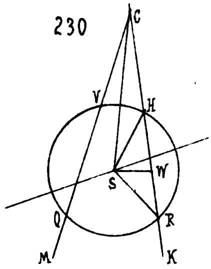

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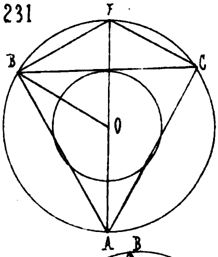

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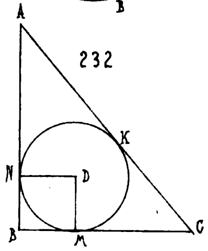

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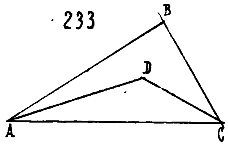

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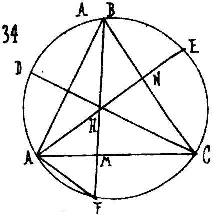

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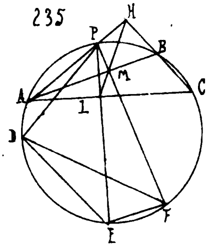

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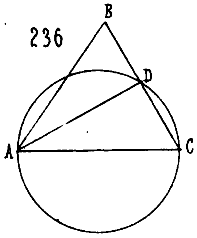

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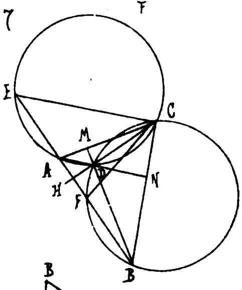

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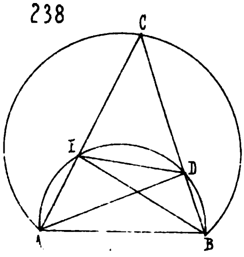

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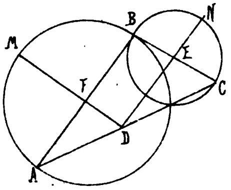

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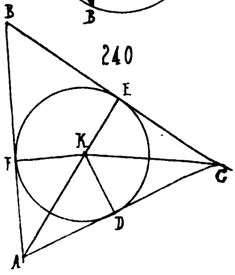

the given difference; from F and G draw FK, GK perpendiculars to AD, AE; describe a circle from K, as centre, and with radius equal to KF or KG; through H draw the tangent $BHNC$ to this circle; then ABC is the triangle required.

Because BF and BN are tangents, they are equal (III. 17 and 36); for the same reason $CN = CG$; and $\therefore$ $BF + CG = BN + NC = BC$; and hence $BA + AC = BC + AF + AG$—that is, the difference between $BA + AC$, and BC is equal to $AF + AG$ = the given difference; hence BAC is the triangle required.

EXERCISE 243.

A circle touches the side BC of a triangle ABC and the other two sides produced: show the distance between the points of contact of the side BC with this circle and with the inscribed circle is equal to the difference between the sides AB, AC.

(Fig. 243, Plate XVII.)—Let the excribed circle touch the base at N, and AB, AC produced at L, K respectively, and let the inscribed circle touch these at M, R, S respectively: $AB = AC + NM$.

Because $AL = AK$, and $AR = AS$, $\therefore$ $RL = SK$—that is, $BL + BM = SC + CN$, and by taking NM away $BL + BN = CM + CS$, or $BN = CM$, $\therefore$ $CS + NM = BM$ or BR; add the equals AR, AS, and $AB = AC + NM$.

EXERCISE 244.

With any point S in the circumference of a given circle, describe a circle cutting the former at K, H; from K draw in the latter circle a chord KZ equal to its radius, and join HZ cutting the given circle at D: show that ZD is equal to the radius of the given circle.

(Fig. 244, Plate XVII.)—Join SZ, SK, SD, DK.

Because $HS = KS$ the angles at D are equal (III. 29 and 27), and since DS is drawn through the centre of the circle HZK, and DZ, DK are drawn making equal angles with it, $\therefore$ $DZ = DK$ (III. 8), $\therefore$ ZDK is isosceles, and ZKS is equilateral, and hence DS bisects the angle ZSK, and because DSK equals half the angle of an equilateral triangle, $\therefore$ DK or DZ is a side of the inscribed hexagon, and equals the radius of the given circle.

EXERCISE 245.

If from any point H in the circumference of a given circle straight lines be drawn to the four angular points of an inscribed square, ABCD, the sum of the squares on the four straight lines is double the square on the diameter.

(Fig. 245, Plate XVII.)—Join AC, BD.

Because $ABCD$ is a square inscribed in a circle, AC, BD are diameters (**IV. 6**), $\therefore$ BHD, AHC are right angles, hence $BH^2 + HD^2 = BD^2$, and $CH^2 + AH^2 = AC^2$—that is, $BH^2 + HD^2 + CH^2 + AH^2 = 2AC^2$, or $2BD^2$.

EXERCISE 246.

Draw from the obtuse angle A of a given triangle to the base a line, the square on which shall be equal to the rectangle under the segments of the base.

(Fig. 246, Plate XVII.)—Circumscribe the given triangle ABC by a circle (**IV. 5**) whose centre is Q; join AQ, and with AQ as diameter describe a circle cutting BC in H.

Join AH: AH is the line required.

For, produce AH to meet the circumference at L, then $CH \cdot HB = AH \cdot HL$ (**III. 35**), but $AH = HL$ (*Ex.* 4); $\therefore$ $CH \cdot HB = AH^2$.

EXERCISE 247.

If the side of a hexagon inscribed in a circle be produced to equal the side of the inscribed square, a tangent to the circle from the extremity of the produced line will be equal to the side of the inscribed octagon.

(Fig. 247, Plate XVII.)—Let ML, MW, MS be the sides of the octagon hexagon, and square respectively; let MW be produced to H so that $MH = MS$; let HQ be the tangent touching the circle at Q.

$HQ = ML$.

From O, the centre of the circle, draw OM and OL cutting MS at K, and in MS take $MZ = MW$ or MO the side of the inscribed hexagon. ZS is evidently equal to WH, and $KL = KZ$.

$MZ^2 + ZS^2 = 2MK^2 + 2KZ^2$ (**II. 9**), but $2MK^2 = MO^2 = MZ^2$, $\therefore$ $ZS^2 = 2KZ^2$.

Now, $MS \cdot SZ = MZ \cdot ZS + ZS^2 = MZ \cdot ZS + 2KZ^2 = MK^2 + KZ^2 = MK^2 + KL^2 = ML^2$, but $MS \cdot SZ = MH \cdot HW$, $\therefore$ $ML^2 = MH \cdot HW = HQ^2$, and hence $HQ = ML$

EXERCISE 248.

A regular octagon inscribed in a circle is equal to the rectangle under the sides of the inscribed and circumscribed square.

(**Fig. 248, Plate XVII.**)—Let *EF* be the side of a square inscribed in the circle *DFEN*, *GH* the side of the circumscribed square, and *FM* the side of the regular octagon inscribed in the circle; then the area of the octagon is equal to the rectangle *GH·FE.*

The rectangle *FE·CM* = twice the area of the quadrilateral *CFME* (I. 41); and ∴ the rectangle *FE·2CM* = four times the area of *CFME* = the area of the regular octagon inscribed in the given circle.

But *FE·2CM* = *FE·DE* = *FE·GH* = the rectangle under the sides of the inscribed and circumscribed squares.

EXERCISE 249.

In Euclid's construction of IV. 10, *suppose the two circles to cut again at E: show that DE is equal to DC; and if AE and BD be produced to meet at F, show that AFB is another isosceles triangle having each of the angles at A and B double the angle at F.*

(**Fig. 249, Plate XVII.**)—Firstly: Because the figure *ACDE* is a quadrilateral inscribed in the smaller circle, the angles *ACD* and *AED* are together equal to two right angles (III. 22); but the angles *BCD* and *ACD* are also (I. 13) equal to two right angles, and ∴ (*Ax.* 3) the angle *BCD* = angle *AED.* But *BCD* = *ABD* = *ADB*; hence the angle *ABD* = angle *AED* = *ADE* (I. 5), and the angle *BAD* = angle *DAE*, and ∴ (III. 26 and 28) *CD* = *DE.*

Secondly: Because the angle *ABF* = twice angle *BAD*, and that angle *BAD* = angle *DAE*, therefore the angle *ABF* = angle *BAF*; and ∴ each of the angles *ABF* and *BAF* is double of the angle *AFB.*

Note.—If *C*, *E* be joined, then *CDFE* is a parallelogram. The angle *ADC* = angle *AEC* (III. 27); the angle *BAD* = angle *ADC* has been shown to be equal to the angle at *F*, ∴ *CD* is parallel to *EF* (I. 28); and since *CDB* = *AEC*, then *AEC* = angle at *F*, and ∴ (I. 28) *CE* is parallel to *DF*, and hence the figure *CDFE* is a parallelogram.

EXERCISE 250.

In (**IV. 10**) *if H be the centre of the larger circle, and HWS the triangle required, and if the side HS cuts the smaller circle in K, and WK produced meets the larger circle at N, and NS be joined: then NK, NS respectively are equal to the sides of a hexagon and pentagon inscribed in the larger circle, and NW is cut as in* (**II. 11**).

(**Fig. 250, Plate XVII.**)—NHK is double of NWS (**III. 20**) and SKW—that is, NKH is double of NWS (**IV. 10**), $\therefore$ $NHK = NKH$, $\therefore$ NK equals NH (**I. 6**), or the side of a regular hexagon inscribed in the larger circle.

Again, $NHS = SKW =$ two-fifths of two right angles, or one-fifth of four right angles, $\therefore$ NS equals the side of a regular pentagon inscribed in NSW; and because $NK = SH$, and $KW = KH$, the line NK is produced to W making KW equal to its greater segment, if it were cut in medial section, $\therefore$ $NW \cdot WK = NK^2$ (*Ex.* 275, PART I).

Note.—The square on the side of a regular pentagon inscribed in a circle is equal to the sum of the squares on the sides of a regular hexagon and a regular decagon inscribed in the same circle.

(**Fig. 250, Plate XVII.**)—It has been shown that NS is the side of a regular pentagon, and NK or NH the side of a regular hexagon inscribed in the circle; and as the angle SHW is one-fifth of two right angles (**IV. 10**), it is one-tenth of four right angles, and $\therefore$ SW is the side of a regular decagon in the same circle; hence it is required to prove that $NS^2 = NH^2 + SW^2$.

From N draw NT perpendicular to HK: then, since $NH = NK$, $KT = HT$ (*Ex.* 10, PART I).

Now, in the triangle SNH, $NS^2 = NH^2 + HS^2 - 2HS \cdot HT$ (**II. 13**) $= NH^2 + HS^2 - HS \cdot HK$ (since $HK =$ twice HT) $= NH^2 + HS\,(HS - HK) = NH^2 + HS \cdot SK$. But $HS \cdot SK = KH^2 = SW^2$; and $\therefore$ $NS^2 = NH^2 + SW^2$.

EXERCISE 251.

On a given straight line AB as base describe an isosceles triangle having the third angle treble of each of the angles at the base.

(**Fig. 251, Plate XVII.**)—Cut AB in C making $AB \cdot BC$

$= AC^2$ (II. 11), and construct the isosceles triangle ADB, having each of the equal sides equal to AC; then ADB is the triangle required.

For the angle at A equals one-fifth of two right angles (IV. 10), and since the angles at A, B are equal, $\therefore$ the angle at D equals three-fifths of two right angles, and is $\therefore$ treble of either of those at A, B.

EXERCISE 252.

If the alternate angles of a regular pentagon be joined (**a**), *the figure formed by the intersection of the joining lines will itself be a regular pentagon*; (**b**) *the greater segment of each is equal to a side of the pentagon*; (**c**) *the diagonals of a regular pentagon which meet within the figure divide one another in medial section*; (**d**) *and the greater segment of each diagonal is also divided in medial section.*

(**Fig. 252, Plate XVII.**)—Let $ABCDE$ be a regular pentagon, and let its alternate angles be joined; then (**a**) the figure $HFGMK$, formed by the intersection of the joining lines, is also a regular pentagon. The angle BCF = angle CBF = angle $DCG = CDG$ (III. 27), and the side $BC = DC$, $\therefore$ $BF = CF = CG = DG$ (I. 6 and 26), and the angle $BFC = DGC = HFG = MGF$ (I. 15). In the same way it may be shown that $AH = CF = BF = GD$; and $\therefore$ (I. 26) $FG = FH = HK = KM = MG$. And as it has been shown that the angle HFG = angle FGM, it can be similarly shown that the angle $FGM = GMK = KHF$; and $\therefore$ the figure $HFGMK$ is a regular pentagon.

(**b**) That $BC = CH = AF$ is evident; for the angle $CHB = HBA + HAB = CBF + FBH$, or the angle $CHB = CBH$, and the side $BC = HC$. Similarly $AF = AB = AM = KD = GE = MC = BG$.

(**c**) Because the angle $CBF = BAF$ (III. 27), BC will be a tangent to the circle circumscribing the triangle ABF (III. 32, *Cor.*); and $\therefore$ the rectangle $AC{\cdot}CF = BC^2$ (III. 36) $= HC^2 = AF^2$. The other diagonals are similarly divided.

(**d**) FD is divided in the medial section in G; for since the angle $DFC = DCF$, and each of them is double of the angle CDF, and that CG bisects the angle DCF, the rectangle $DF{\cdot}FG = GD^2$ (as in IV. 10). Similarly $BG{\cdot}GF = BF^2$.

EXERCISE 253.

To divide a right angle into five equal parts.

(Fig. 253, Plate XVII.)—Let ACB be a right angle; it is required to divide it into five equal parts.

Describe the isosceles triangle CBD as in **IV. 10**, and bisect the angle BCD. Then **(I. 23)** make the angles DCF and FCG each equal to BCE or ECD, each of which is one-fifth of a right angle; the remaining angle GCA is also one-fifth of a right angle.

EXERCISE 254.

If on the sides of a triangle segments of circles be described similar to a segment on the base, and from the extremities of the base tangents be drawn intersecting their circumferences; the points of intersection and the vertex of the triangle will be in the same straight line.

(Fig. 254, Plate XVII.)—Let the segments AKB, CHB be similar to the segment AQC on the base AC of triangle ABC, and let AK, CH be tangents to AQC meeting the circumferences at K, H : K, B, H are in the same straight line.

Join BK, BH.

Because AK, CH are tangents, the angles KAC, HCA are equal, but the angles at K, H are equal **(III.** *Def.* **11)**, $\therefore$ $AKB + KAC =$ two right angles, and hence KB, AC are parallel **(I. 28)**; for a similar reason AC, BH are parallel, $\therefore$ K, B, H are in the same straight line.

EXERCISE 255.

A circle is described round the triangle ABC; the tangent at C meets AB produced at D; the circle whose centre is D and radius DC cuts AB at E: show that EC bisects the angle ACB.

(Fig. 255, Plate XVII.)—Produce DC to H.

Because DC, DE are radii, $DEC = DCE$, $\therefore$ $AEC = HCE$, and since $AEC = EBC + ECB$, and $HCE = EBC + ECA$, $\therefore$ $ECB = ECA$.

EXERCISE 256.

The angle ACB of any triangle is bisected, and the base AB is bisected at right angles by straight lines which intersect at D: show that the angles ACB and ADB are together equal to two right angles.

(Fig. 256, Plate XVIII.)—Describe a circle about the triangle *ACB* (IV. 5); and join *AD* and *DB*.

The line *CD*, which bisects the angle *ACB*, also bisects the arc *ADB* (III. 26); and the line *ED*, which bisects *AB* at right angles, also bisects the arc *ADB* (III. 30); and ∴ the point *D* is on the circumference of the circle circumscribing the triangle *ACB*, and the angles *ACB* and *ADB* are (III. 22) equal to two right angles.

Note.—If $AC = CB$, then *CD* will be a diameter, and will pass through the point *E* (III. 1 and 3).

EXERCISE 257.

If from a point in the produced diameter of a circle a tangent be drawn, and the point of contact be joined with the extremities of the diameter so as to form a triangle, the bisector of the angle between the tangent and produced diameter will cut off from this triangle an isosceles triangle.

(Fig. 257, Plate XVIII.)—Let *CD* be the tangent from the diameter *AB* produced to *C*, and *CKH* the bisector of the angle at *C* meeting the sides *DB*, *DA* of the triangle *ADB* in *K*, *H* respectively: *HDK* is an isosceles triangle.

$CDK = CAH$ (III. 32), and $DCK = ACH$ (*Cons.*). ∴ *DKC* or $HKB = AHC$ (I. 32), ∴ $DHK = DKH$, and hence $DK = DH$, ∴ *HDK* is an isosceles triangle.

EXERCISE 258.

In any triangle ABC, if AD, bisecting the angle A, cut BC at D, and from O, the centre of the inscribed circle, OE be drawn perpendicular to BC: show that the angles BOE, COD are equal.

(Fig. 258, Plate XVIII.)—Draw *AH* perpendicular to *BC*. $DOE = DAH$ equals half the difference of the base angles of *ABC*, or the difference between *OBE*, *OCE* (*Ex.* 103, Part I.), ∴ $OBE = OCE + DOE$, but $OBE + BOE = COE + EOC$, because the angles at *E* are right; and since $OBE = OCE + DOE$, ∴ $BOE = COD$.

EXERCISE 259.

If a circle be described about a triangle ABC, and perpendiculars be let fall from the angular points A, B, C, on the opposite sides, and produced to meet the circle in D, E, F, respectively, the arcs EF, FD, DE are bisected in the points A, B, C.

(Fig. 259, Plate XVIII.)—Let CF, BE cut AB, AC in H, K respectively.

Triangles ABK, ACH are equiangular, $\therefore$ the angles ABK, ACH are equal, and hence the arcs AE, AF are equal (**III. 26**).

It can be similarly shown that the arcs FD, DE are bisected.

EXERCISE 260.

D is the centre of the circle inscribed in a triangle BAC, and AD is produced to meet the straight line drawn through B at right angles to BD at O: show that O is the centre of the circle which touches the side BC and the sides AB and AC produced.

(Fig. 260, Plate XVIII.)—From O let fall the perpendiculars OG and OK on AB and AC produced, and OH on BC; and join OB and OC.

Because ADO passes through the centre O of the inscribed circle, the angle at A is bisected (**IV. 4**, *Cor.*); the angles at G and K are (*Cons.*) right angles, and the line AO is common to the two triangles AOG, AOK, $\therefore$ $OG = OK$ (**I. 26**). Again, since the line BD joins the centre D with the angular point B, the angle EBD = angle DBN; and since (*hyp.*) the angle DBO is a right angle, the angle HBG is bisected by OB (**I. 13**); and since the angles at G and H are right angles, and OB is common, the perpendicular OH = perpendicular OG (**I. 26**). But OK has been shown to be equal to OG, $\therefore$ $OK = OH = OG$ (*Ax.* 1), and hence (**III. 9**), O is the centre of the circle touching the side BC and the sides AB and AC produced.

EXERCISE 261.

In any triangle the line joining the vertex with the centre of the inscribed circle will pass through the centre of the circle passing through the extremities of the base and the centre of inscribed circle.

(Fig. 261, Plate XVIII.)—Let K be the centre of inscibed circle of triangle ABC, and let BK produced meet the circle passing through AKC, at S: KS is a diameter of AKC.

Draw KH perpendicular to AC.

$AKS = CKH$ (*Ex.* 258), and $KSA = KCH$ (**III. 21**), ∴ $KAS = KHC$ (**I. 32**), and is ∴ a right angle, and hence KS is a diameter, and ∴ passes through the centre.

EXERCISE 262.

The opposite sides BA, KS, *and* BK, AS *of a quadrilateral inscribed in a circle are produced to meet at* P, Q *respectively, and about triangles* ASP, KSQ, *circles are described meeting again at* R *: show that* P, R, Q *are in one straight line.*

(Fig. 262, Plate XVIII.)—Join RP, RS, RQ.

Because $ASRP$, $KSRQ$ are quadrilaterals inscribed in circles, $BAS = SRP$, and $BKS = SRQ$, but BAS, BKS equal two right angles (**III. 22**), ∴ SRP, SRQ are equal two right angles, and hence P, R, Q are in the same straight line (**I. 14**).

EXERCISE 263.

O *is the centre of the circle circumscribing a triangle* ABC; D, E, F *the feet of the perpendiculars from* A, B, C *on the opposite sides : show that* AO, OB, OC *are respectively perpendicular to* EF, FD, DE.

(Fig. 263, Plate XVIII.)—Let OA or OH produced meet EF in M.

Because OA, OC, OB are equal, ∴ $EAO + OBA + OCB$ equals half the sum of the angles of the given triangle, and hence equals a right angle; but since $OBC = OCB$, ∴ $EAO + ABC$ equals one right angle, and since the quadrilateral $EFBC$ is circumscribable by a circle, because the angles at F and E are right angles, $AEM = ABC$, ∴ $EAO + AEM$ equals one right angle, and hence AME is a right angle (**I. 32**). In the same way it may be proved of OB, OC.

EXERCISE 264.

AC *is a chord of a given circle*; B *and* D *are two given points in the chord, both within or without the circle : if a circle be described to pass through* B *and* D, *and touch the given circle at* H, *show that* AB *and* CD *subtend equal angles at* H.

(Fig. 264, Plate XVIII.)—Let HA, HC cut the circle in L, S respectively; join LS, and draw the tangent MH.

MHA = *HSL* = *HCA* (III. 32), ∴ *LS* is parallel to *AC*, and ∴ the arcs *LB*, *SD* are equal, hence the angles *BHL*, *DHS* are equal (III. 27). If the points be taken without the circle it can be proved in a similar manner.

EXERCISE 265.

A and B are two points within a circle: find a point P in the circumference such that if PAH, PBK be drawn meeting the circumference at H and K, the chord HK shall be the greatest possible.

(Fig. 265, Plate XVIII.)—Let *APB* be the circle passing through *A*, *B*, and touching the given circle at *P*: then *P* is the point required.

For, since *APB* is the greatest possible angle that *AB* can subtend at the circumference, the arc *HK* is the greatest possible, and hence the chord *HK* is a maximum (III. 26 and 29).

EXERCISE 266.

If through any point M in the circumference of a circle any three chords MD, MK, MS be drawn, and on each, as diameter, a circle be described; these three circles will intersect in three points, which are in the same straight line.

(Fig. 266, Plate XVIII.)—Let the circles intersect in the points *W*, *R*, *H*: then *WRH* is a straight line.

Join *SD*, *SK*, *KD*, and also *MW*, *MR*, *MH*.

Now, by (*Ex.* 10) *SK* produced passes through *H*, and *MH* is perpendicular to it; for a similar reason *KD* produced passes through *W*, and *MW* is perpendicular to it, and *SD* produced passes through *R* in a similar manner, ∴ *W*, *R*, *H* are the points in which perpendiculars from any point *M* on the circumference of the given circle meet the sides of the inscribed *KDS*, and hence (*Ex.* 192) *WHR* is a straight line.

EXERCISE 267.

ABCD is a rectangle; if in triangle ABC a circle be inscribed touching AB, BC at E, F, and EGH, FGK be drawn parallel to AD, CD, then the rectangle KH is equal to the gnomon AFH.

(Fig. 267, Plate XVIII.)—Let *EGH*, *FGK* cut *AC* in *R*, *Q*, and let the circle touch it at *W*.

Join GW.

G is the centre of the circle (III. 19), ∴ the lines GW, AK, and CH are equal, hence triangles AKQ, GWQ are equal, as also are triangles CHR, GWR (I. 26), ∴ the whole triangle QRG equals the sum of triangles AKQ, CHR, and since AC bisects the whole rectangle, ∴ KH = the gnomon AFH.

EXERCISE 268.

If all the sides of a quadrilateral or polygon except one be given in magnitude, the figure will be a maximum when that remaining side is the diameter of a semicircle, and the figure is inscribed in that semicircle.

(Fig. 268, Plate XVIII.)—Let $ABCDE$ be the greatest polygon that can be contained by sides AB, BC, CD, DE given in magnitude, and AE not given: AE is the diameter of a semicircle, and the angular points of the polygon are on the arc of that semicircle.

Join BE.

Then, if $ABCDE$ be not a semicirle, and, therefore, the angle ABE be not a right angle, by making it a right angle the triangle would be enlarged (*Ex.* 119, Part I.), and ∴ the whole polygon would be enlarged.

If the polygon be a maximum the angle ABE must, therefore, be a right angle, and the segment $ABCDE$ a semicircle. In the same way it may be shown that the polygon would be enlarged when ADE or ACE is a right angle, whatever be the number of sides; and ∴ the polygon is a maximum when AE —viz., the side which is not given in magnitude, is the diameter of a semicircle, and the polygon is inscribed in that semicircle.

EXERCISE 269.

Given all the sides of a polygon; its area is a maximum when it is capable of being inscribed in a circle.

(Fig. 269, Plate XVIII.)—In the rectilineal figures $ABCDE$ and $FGHKM$, let $AB = FG$, $BC = GH$, $CD = HK$, $DE = KM$, and $EA = MF$, and let $ABCDE$ be inscribed in a circle, but $FGHKM$ not be capable of being inscribed in a circle: then $ABCDE$ is greater than $FGHKM$.

Draw the diameter BN; join EN and DN; construct the triangle MSK equal to the triangle END, and join GS.

The figure *BAEN* is greater than *GFMS* (*Ex.* 268), and the figure *BNDC* is greater than *GSKH*, and ∴ the entire figure *BAENDC* is greater than *GFMSKH*. But the triangle *END* = triangle *MSK* (*Cons.*), and ∴ (*Ax.* 3) the figure *AEDCB* is greater than the figure *FMKHG*.

EXERCISE 270.

The area of an inscribed regular hexagon is three-fourths that of the one circumscribed about the same circle.

(**Fig. 270, Plate XVIII.**)—Let *BD* be the side of a regular hexagon inscribed in the circle, and *AF* or *AG* the side of a regular circumscribed hexagon.

Join *C*, the centre of the circle, and *A*, *B*, *D*, *G*; and at *D*, with the line *BD*, make the angle *BDW* equal to the angle *ADB* (**I. 23**).

Because the angle *ADB* = the angle *BDW* (*Cons.*), and the angles at *S* are right angles, and *DS* common, the side *AS* = *SW* (**I. 6**); and *DW* = *AD*. Again, because the triangle *ACD* is half of an equilateral triangle, and the angle *ADC* a right angle, the angle *DAC* is the angle of an equilateral triangle, and ∴ since *AD* = *DW*, *ADW* is an equilateral triangle.

But the angle *DCW* = one-third of a right angle; and, since *CDA* is a right angle, and *WDA* = two-thirds of a right angle, the angle *CDW* = one-third of a right angle (*Ax.* 3), and is ∴ equal to the angle *DCW*, and the side *DW* = *CW* (**I. 6**) = *AW*. And since *AS* = *SW*, ∴ *SC* = three-fourths of *AC*, and the triangle *DCS* = three-fourths of the triangle *DCA*. But the inscribed hexagon is twelve times *DCS*, and the circumscribed hexagon twelve times *DCA*, ∴ the inscribed is three-fourths of the circumscribed hexagon.

It can be neatly done by the **VI. 19**.

Note.—This proposition can be proved by various other methods; but that here adopted is, perhaps, the simplest.

PLATE XVII.

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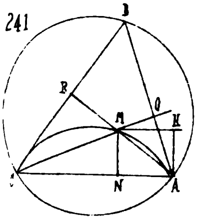

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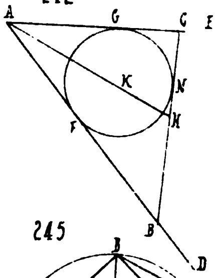

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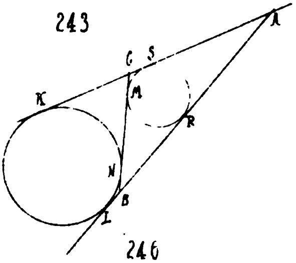

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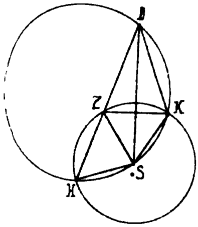

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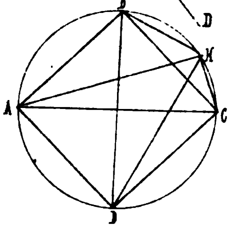

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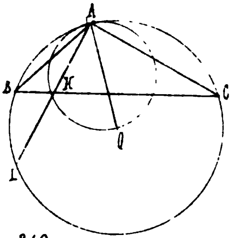

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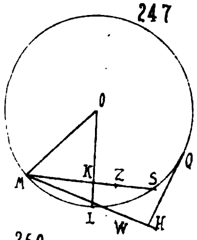

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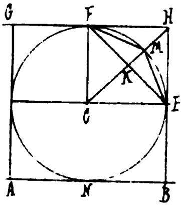

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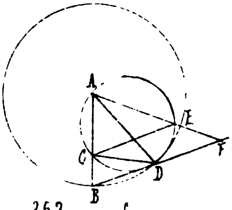

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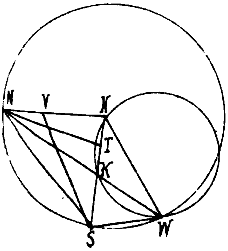

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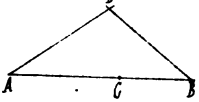

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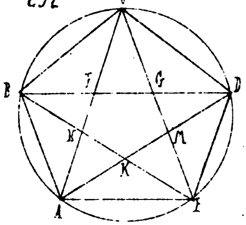

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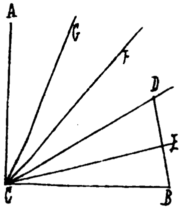

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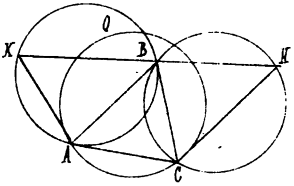

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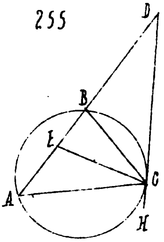

PLATE XVIII

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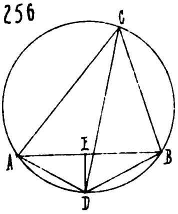

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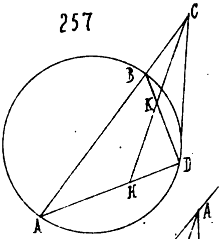

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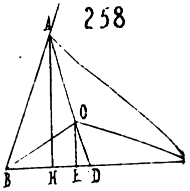

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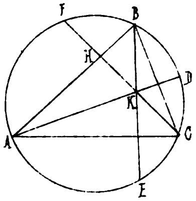

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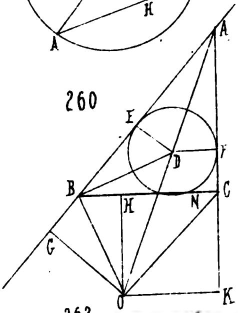

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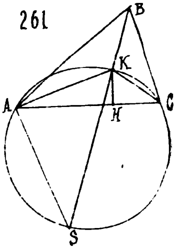

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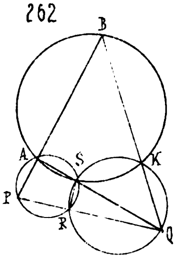

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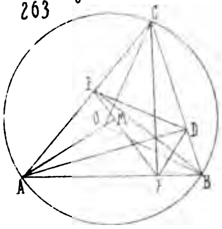

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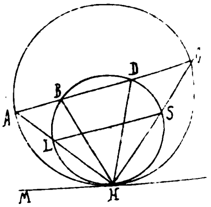

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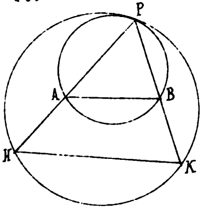

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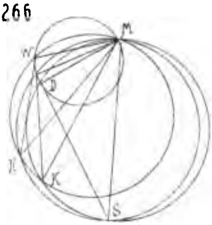

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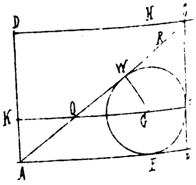

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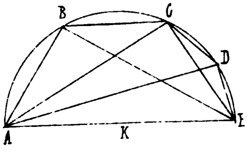

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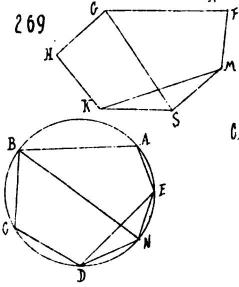

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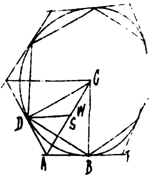


Sixth Book.

EXERCISES UPON EUCLID.

SIXTH BOOK.

EXERCISE 271.

The side BC of a triangle ABC is bisected at D and the angles ADB, ADC are bisected by the straight lines DE, DF, meeting AB, AC at E, F, respectively: show that EF is parallel to BC.

(Fig. 271, Plate XIX.)—Because *DE*, the bisector of the angle *ADB*, meets the base at *E*, $AD : DB :: AE : EB$ (VI. 3), for a similar reason $AD : DC :: AF : FC$, but $DB = DC$ (*Cons.*), $\therefore AE : EB :: AF : FC$, and hence *EF* is parallel to *BC* (VI. 2).

EXERCISE 272.

From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G: show that FG is parallel to CD.

(Fig. 272, Plate XIX.)—$CF : FB :: AE : EB :: DG : GB$ (VI. 2), $\therefore$ *FG* is parallel to *CD*.

EXERCISE 273.

If two triangles be on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.

(Fig. 273, Plate XIX.)—Let the two triangles *ABC*, *DEF* be on equal bases *AC*, *DF*, and between the same parallels *ACDF*, *BE*, and let *WMHS* be drawn parallel to either of the former parallels cutting *AB*, *BC* at *W*, *M*, and *DE*, *EF* at *H*, *S* respectively: then $WBM = HES$.

Because *AF*, *WS*, *BE* are parallel, and (VI. 4) $AC : WM :: AB : BW :: FE : ES :: DF : HS$, but $AC = DF$, $\therefore WM = HS$, and hence $WBM = HES$ (I. 38).

EXERCISE 274.

Apply **VI. 3** *to solve the problem of the trisection of a straight line.*

(Fig. 274, Plate XIX.)—On the given straight line AB construct a triangle ABC, having $AC = AB$, and BC = half AB; now the line CD which bisects the angle ACB cuts AB at D in a point of trisection.

For $AC:CB::AD:DB$ (**VI. 3**), but AC is double of CB, $\therefore AD$ is double of DB.

EXERCISE 275.

Show that the diagonals of a quadrilateral $ABCD$, having AD parallel to BC and double of it, cut one another at F at a point of trisection.

(Fig. 275, Plate XIX.)—Triangles FBC, FDA are equiangular, $\therefore AD:DF::BC:BF$, but AD is double of BC (*Cons.*), $\therefore DF$ is double of BF; for a similar reason AF is double of FC.

EXERCISE 276.

ABC is a triangle; it is required to draw from a given point P in the side AB, or AB produced, a straight line to AC or AC produced, so that it may be bisected by BC.

(Fig. 276, Plate XIX.)—Let P be in AB.

Draw PH parallel to BC meeting AC at H, and produce AC to M making $CM = CH$; draw PM, cutting BC at N: PM is the line required.

For, since NC is parallel to PH, $HC:CM::PN:NM$ (**VI. 2**), but $HC = CM$, $\therefore PN = NM$, or simply because CN is parallel to PH from the middle point of HM, it bisects PM. *Vide* (*Ex.* 133, PART I.).

OTHERWISE:—Draw PR parallel to AC meeting BC at R; produce AC to M, making $CM = CH$, and join PM cutting BC at N; then PM is bisected at N.

For triangles PRN, MCN are equal (**I. 26**), $\therefore PN = MN$.

EXERCISE 277.

ACB is a triangle, and the side AC is produced to D so that $CD = AC$, and BD is joined: if any straight line drawn parallel

to AB cuts the sides AC, CB in R, H respectively, and if HW be drawn parallel to RD meeting AB at Q, W : $AQ =$

(Fig. 277, Plate XIX.)—Draw CM parallel to AB n ing BD at M, and ∴ bisecting it (VI. 2).

Triangles AQR, CMD are equiangular, ∴ (VI. 4) $RQ : A$ $DM : CM :: BM : CM$, and triangles BWH, BCM are e angular; ∴ $HW : BW :: BM : CM$, hence $RQ : AQ :: H$ BW; but since $QRHW$ is a parallelogram $RQ = HW$, $AQ = BW$.

EXERCISE 278.

ABC is a triangle; any straight line parallel to BC meets at D and AC at E; join BE and CD meeting at F: show the triangle ADF is equal to the triangle AEF.

(Fig. 278, Plate XIX.)—Triangles BEC, BDC are eq (I. 37); from each take the common part BFC, and CFE BFD. Again, $CE : EA :: BD : DA$ (VI. 2), but $CFE : AEF$ $CE : EA$ (VI. 1), and $BDF : ADF :: BD : DA$, ∴ $CFE : AEF$ $BFD : ADF$, but since $CFE = BFD$, ∴ $ADF = AEF$.

EXERCISE 279.

AB and CD are two parallel straight lines; E is the mid point of CD; AC and BE meet at F, and AE and BD meet at show that FG is parallel to AB.

(Fig. 279, Plate XIX.)—Because CE is parallel to A (*hyp.*), $AB : CE :: AF : FC$ (VI. 4); and, for the same reaso because ED is parallel to AB, $AB : ED :: AG : GE$; but C $= ED$ (*hyp.*), and ∴ $AF : FC :: AG : GE$, and by divisi $AC : CF :: AE : EG$; and ∴ FG is parallel to CE (VI. 2 and hence to AB.

EXERCISE 280.

If ABC be a triangle having AC cut in D so that $AC \cdot CD$ AD^2, and let D, B joined: show that one of the triangles is mean proportional between the other two.

(Fig. 280, Plate XIX.)—Because $AC \cdot CD = AD^2$ (*hyp* ∴ $AC : AD :: AD : DC$, but $ABC : ABD :: AC : AD$ (VI. 1), a $ABD : DBC :: AD : DC$, ∴ $ABC : ABD :: ABD : DBC$, a hence ABD is a mean proportional between ABC and DBC.

EXERCISE 281.

An isosceles triangle is described having each of the angles at the base double of the third angle: if the angles at the base be bisected, and the points where the lines bisecting them meet the opposite sides be joined, the triangle will be divided into two parts in the proportion of the base to the side of the triangle.

(**Fig. 281, Plate XIX.**)—Let ABC be the triangle, and CD, BE lines bisecting the base angles; join DE: then the quadrilateral $BDEC$ is to the triangle ADE as AC to BC. Because the angles at B, C are bisected and DE is parallel to BC, the angle EDC = angle DCE (I. 29), and the side $DE = EC$ (I. 6).

The triangle BCE = triangle ADE (I. 4); to each add the triangle BDE, and the triangle BEA = the quadrilateral $BDEC$ (*Ax.* 1). Now, the triangle BEA : triangle BEC :: $AE : EC$ (VI. 1); that is, $BDEC : DAE :: AE : EC :: AE : ED$ (EC being = to ED). But $AE : ED :: AC : CB$; and ∴ $BDEC : DAE :: AC : CB$.

Note.—This may also be proved in the following manner:—By VI. 19 $ABC : ADE :: AB^2 : AD^2 :: AB^2 : AB \cdot BD :: AB : BD$; and, by division, $ABC - ADE : ADE :: AB - BD : BD$; that is, $BDEC : ADE :: AD : DB :: AD : DE :: AB : BC$ (VI. 2).

EXERCISE 282.

Through D, any point in the base of a triangle ABC straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: show that the triangle AEF is a mean proportional between the triangles FBD, EDC.

(**Fig. 282, Plate XIX.**)—$CE : EA :: CD : DB :: AF : FB$ (VI. 2), but $CE : EA :: EDC : AEF$ (VI. 1), and $AF : FB :: AEF : FBD$, ∴ $EDC : AEF :: AEF : FBD$; hence AEF is a mean proportional between FBD and EDC.

EXERCISE 283.

The two sides AB, AC of a given triangle ABC are produced to any points D, E, so that DE is parallel to BC. The straight line DE is divided at F so that DF is to FE as BD is to CE: show that the locus of F is a straight line.

(Fig. 283, Plate XIX.)—Join *AF*. *AD* : *AE* :: *BD* : *CE* :: *DF* : *FE*, hence *AF* bisects the vertical angle (VI. 3), and is the locus required.

EXERCISE 284.

ABC is a triangle, and a perpendicular is drawn from A to the opposite side meeting it at D between B and C : show that if BA is a mean proportional between BD, BC, the angle BAC is a right angle.

(Fig. 284, Plate XIX.)—*BC* : *BA* :: *BA* : *BD* (*hyp.*), ∴ triangles *BCA*, *BDA* are equiangular (VI. 6), but the angle at *D* is right, hence *BAC* is a right angle.

EXERCISE 285.

ABC is a triangle, and a perpendicular is drawn from A to the opposite side meeting it at D between C and B; if AD is a mean proportional between the segments of the base; the angle BAC is a right angle.

(Fig. 285, Plate XIX.)—*DB* : *DA* :: *DA* : *DC* (*hyp.*), and the angles at *D* are right, ∴ triangles *DCA*, *DBA* are equiangular (VI. 6), and those angles are equal which are opposite to homologous sides; hence *DCA* = *DAB*, and *DAC* = *DBA*, ∴ *BAC* = *DCA* + *DBA*, and is ∴ a right angle (I. 32).

EXERCISE 286.

CAB, CEB, are two triangles which have a common angle B, and the sides CA, CE equal: if in BE produced there be taken ED, a third proportional to BA, AC, then will the triangles BDC, BAC be similar.

(Fig. 286, Plate XX.)—Angles *CAB*, *CED* are equal, and *AB* : *AC* :: *EC* : *ED*, ∴ *BCA* = angle *D* (VI. 6), hence *BDC*, *BAC* are similar.

EXERCISE 287.

In the circumference of a circle of which AB is a diameter, take any point P; and draw PC, PD on opposite sides of AP, and equally inclined to it, meeting AB at C and D: show that AC is to BC as AD is to BD.

(Fig. 287, Plate XX.)—Let *C* fall without the circle, and produce *CP* to *M*; join *PB*.

Because *APB* is a right angle it equals the angles *CPA*, *BPM*, but *APD* = *CPA* (*Cons.*), ∴ *DPB* = *BPM*, and hence *BC* : *BD* :: *AC* : *AD* (**VI. 3** and **4**), ∴ alternately and inversely *AC* : *BC* :: *AD* : *BD*.

EXERCISE 288.

AB is a diameter of a circle, CD is a chord at right angles to it, and E is any point in CD; AE and BE are drawn and produced to cut the circle at F and G: show that the quadrilateral CFDG has two of its adjacent sides in the same ratio as the remaining sides.

(Fig. 288, Plate XX.)—Let *CD* cut *AB* at *W*, and *CB*, *DB joined.*

Triangles *CWB*, *DWB* are equal (**III. 3**), and (**I. 4**), ∴ *CB* = *DB*, and hence *CGB* = *DGB* (**III. 28** and **27**), ∴ *CE* : *ED* :: *CG* : *GD* (**VI. 3**), and for a similar reason *CE* : *ED* :: *CF* : *FD*, ∴ *CG* : *GD* :: *CF* : *FD*.

EXERCISE 289.

The line AB is divided at C, D, so that AB : AC :: AC : AD; if AE be another line taken equal to AC, show that the angle BED is bisected by EC.

(Fig. 289, Plate XX.)—Because (*hyp.*) *BA* : *AC* :: *AC* : *AD*; that is, *BA* : *AE* :: *AE* : *AD*, the triangles *ABE* and *ADE* are (**VI. 6**) similar, and the angle *AED* = angle *ABE*. But since (*hyp.*) *AE* = *AC*, the angle *ACE* = the angle *AEC*; but the angle *ACE* = the angles *EBC* and *BEC* (**I. 32**); ∴ the angle *AEC* = the angles *EBC* and *BEC*. But the angle *AED* = the angle *EBC*, ∴ the remaining angles *DEC* and *BEC* are equal (*Ax.* 3), and hence *EC* bisects the angle *BED*.

EXERCISE 290.

If two points be taken in the diameter of a circle, such that the rectangle contained by the segments intercepted between them and the centre of the circle be equal to the square of the radius; and if from these points straight lines be drawn to any point in the circum-

ference, the ratio of these lines will be equal to the ratio of the segments intercepted between the two first-mentioned points and the circumference of the circle, and the included angle will be bisected by the straight line joining the two points in the circumference.

(Fig. 290, Plate XX.)—Let *CED* be a circle, whose centre is *K*, and in *KC* produced, let the points *B* and *A* be taken such, that the rectangle $AK \cdot KB$ is equal to the square on *CK*; then if straight lines be drawn from *B* and *A*, to any point *E* in the circumference, $AE : EB :: AC : CB$, and the the angle *AEB* is (VI. 3) bisected by *EC*.

Join *KE*.

Then because $AK \cdot KB = CK^2 = EK^2$ (*hyp.*), and the angle *AKE* is common to the two triangles *AEK* and *BEK*, these triangles are similar (VI. 6), and (VI. 4) $AE : EK :: EB : BK$, and alternately $AE : EB :: EK : KB :: CK : KB$; but (*hyp.*) $AK : CK :: CK : KB$, and by division $AC : CK :: CB : BK$; and alternately $AC : CB :: CK : KB$. But it has been shown that $AE : EB :: CK : KB$; and $\therefore AC : CB :: AE : EB$; and therefore (VI. 3) the angle *AEB* is bisected by the straight line *EC*.

Note.—If *ED* be joined it will bisect the angle *BEF*.

EXERCISE 291.

ABCD is a parallelogram; P, Q are points in a straight line parallel to AB; PA and QB meet at R, and PD, QC meet at S; show that RS is parallel to AD.

(Fig. 291, Plate XX.)—$AB : PQ :: AR : RP$ (VI. 4), and $DC : PQ :: DS : SP$, $\therefore AR : RP :: DS : SP$, hence *RS* is parallel to *AD* (VI. 2).

EXERCISE 292.

ABC is an isosceles triangle, the angle at A being four times either of the others: show that if BC be trisected in D, E the triangle ADE is equilateral.

(Fig. 292, Plate XX.)—$AD = AE$ (I. 4), and the perpendicular *AF* bisects *DE* in *F*, $\therefore$ *CD* is double of *DF*, and since the angle at *F* is right, and *CAF* is double of *ACF*, *AC* is double of *AF*; $\therefore CD : DF :: AC : AF$, hence *AD* bisects the angle *CAF* (VI. 3), $\therefore DCA = DAC$, and $DA = DC$; for a similar reason $EA = EB$, $\therefore$ *ADE* is equilateral.

EXERCISE 293.

AB is the diameter of a circle, E the middle point of the radius OB; on AE, EB as diameters circles are described; PQL is a common tangent meeting the circles at P and Q, and AB produced at L: show that BL is equal to the radius of the smaller circle.

(Fig. 293, Plate XX.)—Join C, the centre of the larger of the two inner circles, and P; join K, the centre of the smaller of the inner circles, and Q.

Then, since the angles at P, Q are right angles (III. 18), and the angle at L common to the two triangles CPL, KQL, these triangles are similar (VI. 4), and $CP : KQ :: CL : KL$; but CP is (*Cons.*) $= 3KQ$, $\therefore CL = 3KL$, and hence $CK = 2KL$. Again (*Cons.*), $CK = 4KB$, $\therefore 4KB = 2KL$, and hence $2KB = KL$; and $\therefore KB = BL =$ the radius of the smaller of the inner circles.

EXERCISE 294.

A and B are two given points; AC and BD are perpendicular to a given straight line CD; AD and BC intersect at E, and EF is perpendicular to CD: show that AF and BF make equal angles with CD.

(Fig. 294, Plate XX.)—In the similar triangles BDE and AEC (I. 29 and VI. 4), $BD : AC :: BE : EC$; and because EF is parallel to BD, $BE : EC :: DF : FC$; and $\therefore$ $BD : AC :: DF : FC$, and alternately $BD : DF :: AC : FC$, and therefore (VI. 6) the triangles BDF and ACF are equiangular, and the angle $BFD =$ angle CFA; that is, AF and BF make equal angles with CD.

EXERCISE 295.

Three points D, E, F in the sides of a triangle ABC being joined form a second triangle, such that DF, DE make equal angles with BC, also ED, EF with AC, and FD, FE with AB: show that AD, BE, CF are at right angles to BC, CA, AB respectively.

(Fig. 295, Plate XX.)—Produce FD to W, and let FC cut DE in M.

Because DC bisects the exterior angle of triangle FDM and meets the base FM produced, $FD : DM :: FC : CM$ (VI. 4), for a similar reason $FE : EM :: FC : CM$, $\therefore FD : DM :: FE : EM$, and hence CF bisects the angle EFD (VI. 3), $\therefore AFC = BFC$, and CF is at right angles to AB. It may be similarly shown that AD, BE are perpendicular to CB, CA respectively.

EXERCISE 296.

Find a point without a given circle such that the sum of the two tangents from it to the circle shall be equal to the secant from it through the centre.

(**Fig. 296, Plate XX.**)—Draw the diameter ACB; from A draw AD at right angles to AB, and equal to half the radius AC; through D draw the tangent FDP meeting BA produced in P: then P is the required point.

Because the triangles CPF and APD have each a common angle at P, and a right angle, they are similar, and $CF : FP :: DA : AP$; and alternately $CF : DA :: FP : AP$. But $CF = 2DA$ (*Cons.*), $\therefore$ $FP = 2AP$. Now, $BP \cdot PA = FP^2$ (**III. 36**) $= 4AP^2$; and hence $BP = 4AP$, and $\therefore$ $BA = 3AP$; that is, AP = one-third of the diameter.

EXERCISE 297.

The straight line AD is divided into three equal parts in the points B and C; a circle is described with B as centre and BA as radius, and any circle cutting this is described with D as centre: show if a chord to both the circles be drawn from A, through one of the points of intersection, it will be bisected by this point.

(**Fig. 297, Plate XX.**)—Let APH be the chord to both circles; then AP is equal to PH.

From B and D, the centres of both circles, let fall perpendiculars BM, DN, which (**III. 3**) bisect AP, PH, respectively; join CP.

Now, since the angle APC is a right angle (**III. 31**), BM, CP, and DN are parallel; and $\therefore$ (**VI. 2**) $AB : BC :: AM : MP$; and since $AB = BC$ (*hyp.*), $AM = MP$. Again, $BC : CD :: MP : PN$; and since $BC = CD$, $MP = PN$. But MP = half of AP, and PN = half of PH; and $\therefore$ $AP = PH$.

EXERCISE 298.

To draw a straight line parallel to a given straight line such that the part of the parallel intercepted between two other straight lines given in position may have a given ratio to the perpendicular distance between the two parallels.

(**Fig. 298, Plate XX.**)—Let AB be a given straight line falling on two other given straight lines AC, BC; it is required to draw a parallel to AB such that the part of it intercepted between AC and BC may have a given ratio to the perpendicular distance between AB and the parallel.

Through C draw CD parallel to AB and CF perpendicular to it; and take CD to CF in the given ratio. Join A, D cutting CB in E; draw EG parallel to AB, and GH parallel to CF or perpendicular to AB; then EG is the line required.

In the similar triangles ACD, AGE, $CD : CA :: EG : GA$; and, alternately, $CD : EG :: CA : GA$. But in the similar triangles ACF, AGH, $CA : GA :: CF : GH$; and, $\therefore$, from equality of ratios, $CD : EG :: CF : GH$; and, alternately, $CD : CF :: EG : GH$. Hence, therefore, EG is to GH in the given ratio; and EG is parallel to AB.

Note.—If CD be equal to CF, EG will be equal to GH; and EG or GH will be a side of an inscribed square.

EXERCISE 299.

If two triangles have one angle of the one equal to one angle of the other and another angle of the one supplementary to another angle of the other, the sides opposite to these four angles will be proportionals.

(**Fig. 299, Plate XX.**)—Let the two triangles ABC and ADE have angles BAC and DAE equal, and let the angles BCA and EDA be supplementary.

Place these triangles so that AC and AD may be in one line; then AB and AE shall also be in one straight line (I. 14). Let the angles BCA be greater than ADE.

From B draw BF parallel to DE. Because BF is parallel to DE, the angles BFC and ADE are equal (I. 29); and because BCA and ADE are supplementary, and BCF and BCA are also supplementary, $\therefore BCF = BFC$, and $BF = BC$. Now, the triangles ABF and ADE being similar, $AB : BF :: AE : ED$; but, since $BF = BC$, this proportion becomes $AB : BC :: AE : ED$.

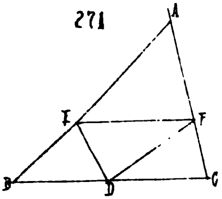
EXERCISE 300.

$ABCD$ is a quadrilateral inscribed in a circle; BA, CD, are produced to meet in P, and AD, BC, are produced to meet in Q: prove that PC is to PB as QA is to QB.

(**Fig. 300, Plate XX.**)—In the two triangles AQB and CPB, since the angle B is common and the angles QAB and PCB supplemental, $CP : PB :: QA : QB$ (*Ex.* 299).

PLATE XIX

271



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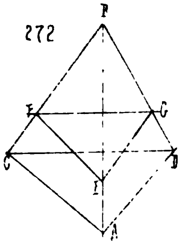

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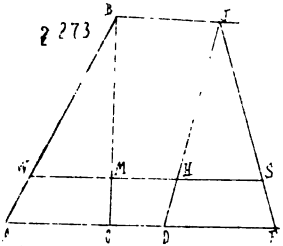

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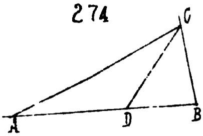

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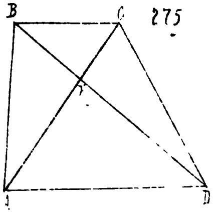

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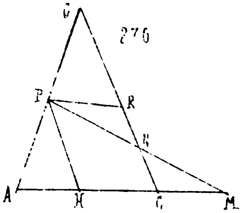

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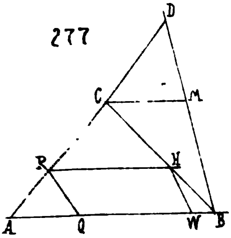

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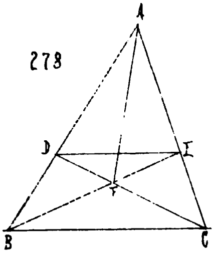

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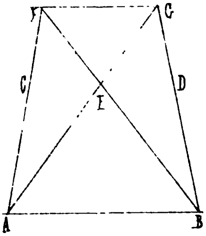

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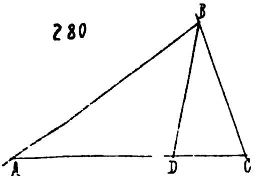

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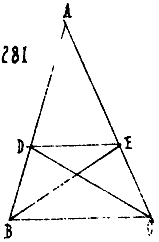

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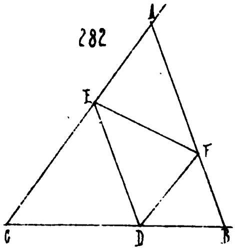

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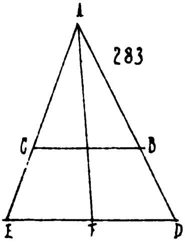

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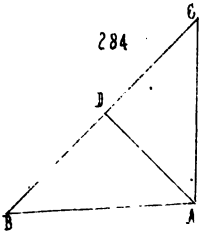

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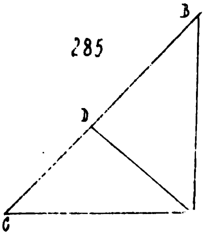
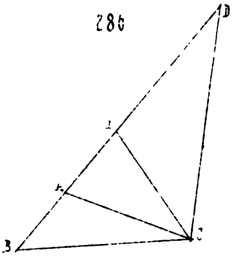


PLATE IX.

286



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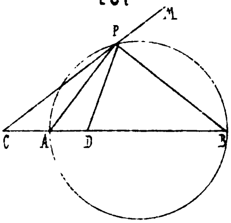

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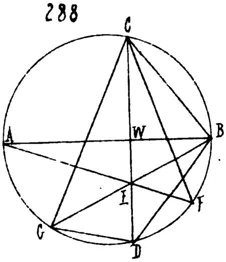

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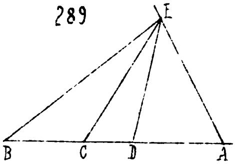

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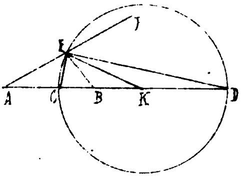

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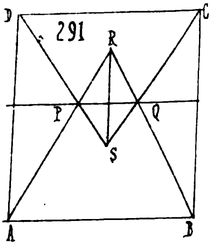

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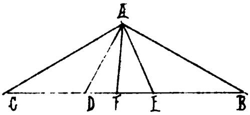

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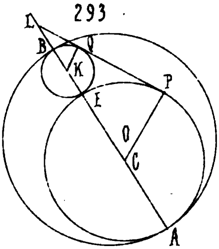

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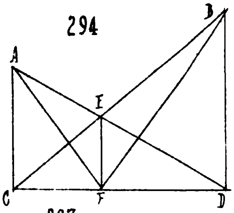

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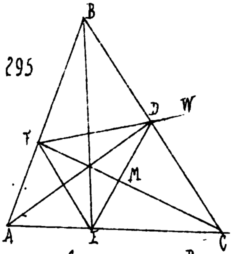

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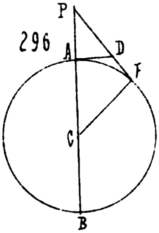

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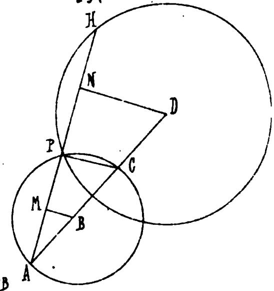

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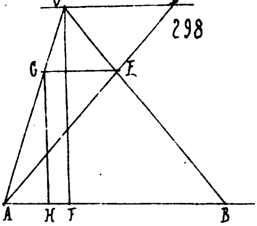

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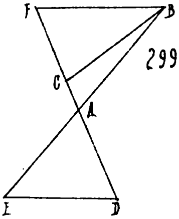

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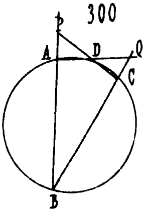

EXERCISE 301.

The angle A of a triangle is bisected by AD which meets the base at D, and O is the middle point of BC: show that OD bears the same ratio to OB that the difference of the sides bears to their sum.

(Fig. 301, Plate XXI.)—Now, because OD equals half the difference of DB and DC, and AD bisects the angle A, $\therefore$ $OD : OB :: AB - AC : AB + AC$.

EXERCISE 302.

In the sides AB, AC of a triangle ABC are taken two points D, E, such that BD is equal to CE; DE BC are produced to meet at F: show that AB is to AC as EF is to DF.

(Fig. 302, Plate XXI.)—From D draw DG parallel to AC. Then **(VI. 4)** $AB : AC :: DB = EC : DG :: EF : DF$.

EXERCISE 303.

If from any angle of a rectangular parallelogram, a line be drawn to the opposite side, and from the adjacent angle of the trapezium thus formed a perpendicular be drawn to the former; the rectangle contained by these two lines, is equal to the given parallelogram.

(Fig. 303, Plate XXI.)—Let E be a point in the side DE of the rectangle $ABCD$, to which AE is drawn, and BF a perpendicular on this line: then $AE \cdot BF =$ the rectangle $ABCD$.

Because the angles at D, F are right, and $AED = EAB$ **(I. 29)**, $\therefore$ triangles DAE, FBA are similar **(VI. 4)**, $\therefore$ $AE : AB :: AD : BF$, hence $AE \cdot BF = AB \cdot AD =$ the given rectangle.

OTHERWISE :—Join BE.

Then $AE.BF =$ twice triangle $AEB =$ rectangle $ABCD$.

EXERCISE 304.

ABC and ABF are triangles on the same base AB in the ratio of two to one; AF and BF produced meet the sides at D and E; in FB, a part FG is cut off equal to FE, and BG is bisected at O: show that BO is to BE as DF is to DA.

(Fig. 304, Plate XXI.)—Join CF, and produce it to meet AB in H; join HG; and through H draw HK parallel to BC.

Because the triangle ABC is double of the triangle ABF (*hyp.*), and they are on the same base, AB, the altitude of ABC is double that of ABF, and CF is equal to FH. But (*hyp.*) $EF = FG$, and the angle EFC = angle GFH (**I. 15**), $\therefore$ the triangles EFC and HFG are equal in every respect (**I. 4**), and the angle ECF = angle GHF, and therefore (**I. 27**) EC or AC is parallel to GH. Again, since HK is parallel to CD, and $CF = FH$, the triangle FCD = triangle FHK, and $DF = KF$. Now, because GH is parallel to AE, $EB : BG :: AB : BH$ (**VI. 2**); and because HK is parallel to BD, $AB : BH :: AD : DK$; and $\therefore AD : DK :: EB : BH$. But, since $BG = 2BO$ (*Cons.*), and $DK = 2DF$, the last proportion becomes $AD : 2DF :: EB : 2BO$; and $\therefore AD : DF :: EB : BO$, and, inversely, $DF : DA :: BO : BE$.

EXERCISE 305.

To describe a parallelogram having its adjacent sides respectively equal to the segments of one straight line divided in extreme and mean ratio, and its diagonals equal to the segments of another straight line divided in the same manner.

(Fig. 305, Plate XXI.)—Divide any straight line AC in extreme and mean ratio in D (**VI. 30**); on AC describe the equilateral triangle ABC; through D draw DE parallel to CB, and DF parallel to AB (**I. 31**): then $DEBF$ is the parallelogram required.

Join BD and EF. Because ED is parallel to BC, the triangle AED is equilateral; and because DF is parallel to AB, the triangle DFC is equilateral; and $DF = EB$, and $DE = BF$ (**I. 34**); and $\therefore AB$ is divided in E, and BC in F, similarly to AC in D. Now (*hyp.*), $CA : AD :: AD : DC$; or $BA : AD :: FB : BE$; and the angles at A and B are (*Cons.*) equal, $\therefore$ the triangles ABD and FBE are similar, and $BE : EF :: AD : DB$; that is, $DC : EF :: AD : DB$, and alternately $DC : AD :: EF : DB$; and by composition $CA : AD :: EF + DB : DB$. But (*Cons.* and *hyp.*) $CA : AD :: AD : DC$; and $\therefore AD : DC :: EF + DB : DB$; but $ED = AD$, and $DF = DC$, hence also DB and EF are the segments of a line equal to their sum cut in extreme and mean ratio.

EXERCISE 306.

In any triangle, right-angled at A, *if* CD *be drawn bisecting the angle* C, *show that* $AB : AC :: BC - AC : AD$.

(Fig. 306, Plate XXI.)—From D draw DH at right angles to BC. Then, since the triangles BCA and BDH are similar, $BA : AC :: BH : HD$; that is, since $HC = AC$, and $HD = AD$ **(I. 26)**, $BA : AC :: BC - AC : AD$.

EXERCISE 307.

$ABCD$ *is a quadrilateral in a circle; the straight lines* CE, DE *which bisect the angles* ACB, ADB *cut* BD *and* AC *at* F *and* G *respectively: show that* EF *is to* EG *as* ED *is to* EC.

(Fig. 307, Plate XXI.)—Because the angles ADB and ACB are equal **(III. 27)**, their halves, the angles EDF and ACE are equal, and the angle DEC is common to the two triangles DEF and CEG, ∴ these two triangles are equiangular **(I. 32)**, and $FE : ED :: GE : EC$ **(VI. 4)**, and alternately $FE : EG :: ED : EC$.

EXERCISE 308.

A tangent to a circle at the point A *intersects two parallel tangents at* B, C, *the points of contact of which with the circle are* D, E *respectively; and* BE, CD *intersect at* F: *show that* AF *is parallel to the tangents* BD, CE.

(Fig. 308, Plate XXI.)—Triangles FDB, FCE are equiangular, ∴ $BF : BD :: EF : EC$, but $BD = BA$ and $CE = CA$, being tangents, ∴ $BF : BA :: EF : CA$, and hence AF is parallel to CE or BD **(VI. 2)**.

EXERCISE 309.

From any point P *tangents* PM, PN *are drawn to a circle, and* MK *is drawn perpendicular to the diameter* HN: *show that* MK *is bisected by* PH *at* S.

(Fig. 309, Plate XXI.)—Produce PM to meet the tangent HW. Bearing **(III. 16)** in mind the following ratios suggest themselves:—$SK : PN :: HS : HP :: WH : WP :: SM : PN$, ∴ $SK = SM$.

Or, thus—Join HM and produce it to meet NP produced at T. Now the angle $PTM = WHP = WMH = PMT$, $\therefore PM = PT$, hence PT, PM are equal, $\therefore HP$, drawn to the middle point of NT, bisects MK drawn parallel to NT (*Ex.* 156, Part I.).

EXERCISE 310.

If two circles touch each other, and also touch a given straight line, the part of the straight line between the points of contact is a mean proportional between the diameters of the circles.

(Fig. 310, Plate XXI.)—Let ACD and BCE be two circles touching in the point C, and let them touch a straight line AB; from A and B draw the diameters AD, BE; join A, D, B, E with C; from C draw CT a common tangent to the circles.

Because AT and CT are tangents from T they are equal, and the angle TAC = angle TCA; for the same reason $TB = TC$, and the angle TBC = angle TCB, $\therefore$ the angle $ACB = TAC + TBC$, and hence ACB is (I. 32) a right angle. Again, the angles ACD and BCE are (III. 31) right angles, and hence ACB and BCE are right angles, and (I. 14) ACE and BCD are straight lines.

The angle BAE = angle ADC (III. 32); for the same reason the angle ABC = angle BE; and the angles BAD and ABE are right angles, $\therefore$ the triangles DBA and ABE are similar, and $DA : AB :: AB : BE$.

Note.—See Exercise 202, in which this is done by III. Book.

*EXERCISE 311.

From the extremities B, C of the base of an isosceles triangle ABC, straight lines CE, BF are drawn at right angles to AB, AC respectively, and intersecting at D: show that the rectangle $BC \cdot AD$ is double of the rectangle $AB \cdot DE$.

(Fig. 311, Plate XXI.)—Produce AD to meet BC at H. Because the angles at E and H are right angles (*hyp.*), and the angle EAH common, the triangles ABH and ADE are similar; and $\therefore AB : BH :: AD : DE$, and hence the rectangle $AB \cdot DE$ = rectangle $AD \cdot BH$. Now $BC = 2BH$, and the rectangle $AD \cdot BC = 2AD \cdot BH$; and $\therefore AD \cdot BC = 2AB \cdot DE$.

Otherwise :—$BH \cdot AD = 2ADB = AB \cdot DE$, and $CH = BH$, $\therefore BC \cdot AD = 2AB \cdot DE$.

EXERCISE 312.

If a rectangle be inscribed in a right-angled triangle, having the right angle common, the rectangle of the segments of the hypotenuse will be equal to the sum of the rectangles of the segments of the sides.

(Fig. 312, Plate XXI.)—Let the rectangle $DEBF$ be inscribed in the right-angled triangle ABC, having the right angle at B common; then the rectangle $AD \cdot DC = AE \cdot EB + CF \cdot FB$.

From E let fall the perpendicular EG. Because the triangles AEG and DFC are similar, $EA : AG :: CD : DF$, and $\therefore EA \cdot DF = EA \cdot EB = AG \cdot CD$. Again, since the triangles DEG and DCF are similar, $ED : DG :: CD : CF$, and $\therefore ED \cdot CF = BF \cdot CF = DG \cdot CD$. Now, adding equals to equals, $EA \cdot EB + BF \cdot CF = AG \cdot CD + DG \cdot CD = (AG + DG) \cdot CD = AD \cdot CD$.

Another Method (without the VI. Book):—

$AC^2 = AB^2 + BC^2$ (I. 47); and as $AC^2 = AD^2 + DC^2 + 2AD \cdot DC$ (II. 4), and $AB^2 = AE^2 + EB^2 + 2AE \cdot EB$, and $BC^2 = BF^2 + FC^2 + 2BF \cdot FC$, hence $AD^2 + DC^2 + 2AD^2 \cdot DC = AE^2 + EB^2 + BF^2 + FC^2 + 2AE \cdot EB + 2BF \cdot FC$. Again, since $AD^2 = AE^2 + ED^2 = AE^2 + BF^2$; and $DC^2 = DF^2 + CF^2 = EB^2 + CF^2$; $\therefore$ (*Ax.* 1) $AE^2 + BF^2 + 2AD \cdot DC + DF^2 + CF^2 = AE^2 + EB^2 + AE \cdot EB + BF^2 + FC^2 + BF \cdot FC$, and (*Ax.* 3) $2AD \cdot DC = 2AE \cdot EB + 2BF \cdot FC$; and $\therefore AD \cdot DC = AE \cdot EB + BF \cdot FC$.

EXERCISE 313.

On the sides BA, BC of a right-angled triangle ABC, right-angled at B, squares $BSDA$, $BHEC$ are described; and CD, AE are drawn cutting BA, BC at N, M respectively: show that equal parts are cut off from the sides of the triangle by these lines, and that each of these equal parts is a mean proportional between the remaining segments of the sides.

(Fig. 313, Plate XXI.)—$SC : CB :: DS : BN$ (VI. 4), that is, $AH : HE :: AB : BN :: AB : BM$, $\therefore BN = BM$.

Again, $AN : BN :: AD : BC$ (VI. 4), that is, $AN : BN :: AB : BH :: BM : CM$, hence BN or BM is a mean proportional between AN, CM.

EXERCISE 314.

ABC is a triangle having a right angle at C; $ABDE$ is the square described on the hypotenuse; F, G, H are the points of intersection of the diagonals of the squares on the hypotenuse and sides: show that the angles DCE, GFH are together equal to a right angle.

(Fig. 314, Plate XXI.)—Join FA, FE.

Triangles AFE, AHC are equiangular, $\therefore AE : AC :: AF : AH$ **(VI. 4)**, and angles EAC, FAH are equal, $\therefore$ triangles EAC, FAH are equiangular **(VI. 6)**, $\therefore AEC = AHF$, for a similar reason $BCD = BGF$, but $GFH = AHF + BGF = ACE + BCD$, hence $DCE + GFH = ACB$, which is a right angle.

EXERCISE 315.

A straight line AB is divided into any two parts at C, and on the whole straight line and on the two parts of it equilateral triangles ADB, ACK, BCF are described, the two latter being on the same side of AB, and the former on the opposite side; G, H, K are the centres of the circles inscribed in these triangles: show that the angles AGH, BGK are respectively equal to the angles ADC, BDC, and that $GH = GK$.

(Fig. 315, Plate XXI.)—Join CH, DG.

$AD : AC :: AG : AH$, because AGD, AHC are equiangular **(VI. 4)**, and the angles DAC, GAH are equal **(IV. 4)** $\therefore$ triangles DAC, GAH are equiangular **(VI. 6)**, hence the angles ADC, AGH are equal. It can be shown in like manner that BDC, BGK are equal.

Again, because $DA : DC :: GA : GH$, and $DB : DC :: GB : GK$, $\therefore GA : GH :: GB : GK$, but $GA = GB$, $\therefore GH = GK$.

EXERCISE 316.

If a straight line touch a circle, and a perpendicular be drawn from the point of contact on any diameter, and if from the extremities of this diameter and from the centre perpendiculars be drawn to meet the tangents; the four perpendiculars will be proportionals.

(Fig. 316, Plate XXII.)—Let the straight line touch the circle at G, of which AH is the diameter, and D the centre; let AC, DE, FG, HM be the perpendiculars to the diameter meeting the touching line at C, E, G, M; produce CM, AH to meet at B, and join DG: then AC, DE, FG, HM are proportionals.

Because DGB is a right angle, $DB \cdot BF = BG^2 = AB \cdot BH$ (III. 36), $\therefore$ $AB : DB :: BF : BH$; that is, $AC : DE :: FG : HM$.

EXERCISE 317.

If a perpendicular be drawn from the right angle to the hypotenuse of a right-angled triangle, and circles be inscribed in the two smaller triangles into which the given triangle is divided, their diameters will be to each other as the sides containing the right angle.

(**Fig. 317, Plate XXII.**)—Let E, D be the centres of the circles inscribed in the triangles BAF, CAF; EM, DN their radii on the sides about the right angle; and let AF be the perpendicular from the right angle A of the given triangle BAC.

Join EA, EB, DA, DC.

The angles FAC, FBA are equal, $\therefore$ angles DAC, EBA are equal (IV. 4); for a similar reason $DCA = EAB$, $\therefore$ DCA, EAB are similar, as also are DCN, EAM, $\therefore$ $2DN : 2EM :: DC : EA :: AC : AB$; that is, the diameters are to each other as the sides about the right angle.

EXERCISE 318.

If through the vertex and the extremities of the base of a triangle two circles be described intersecting each other in the base, or base produced, their diameters are proportional to the sides of the triangle.

(**Fig. 318, Plate XXII.**)—Let the circles ABM, CBM intersect in the point M in the base AC of triangle ABC; and let BR, BK be their respective diameters.

Now, R, M, K, are in the same straight line (*Ex.* 10, PART II.), and angle R = angle A (III. 21), for a like reason the angles at C, K are equal, $\therefore$ triangles ABC, RBK are equiangular (I. 32), hence $BR : BK :: BA : BC$ (VI. 4).

EXERCISE 319.

From the point of intersection of two tangents to a circle let a secant be drawn, and the points of contact joined with the points where the secant cuts the circle; a quadrilateral will be formed in which the rectangle under the opposite sides are equal.

(Fig. 319, Plate XXII.)—Let M be the point of intersection of the tangents MN, MQ, and MWH, the secant; join NM, NH, QW, QH: then $QH \cdot NW = NH.QW$.

Triangles HMN, NMW are similar, $\therefore HN : NW :: MN : MW$; for a similar reason $HQ : QW :: MQ : MW$, but $MN = MQ$, $\therefore HN : NW :: HQ : QW$; hence $HN \cdot QW = NW \cdot HQ$ (VI. 16).

EXERCISE 320.

A and B are two points on the circumference of a circle of which C is the centre; draw tangents at A and B meeting at T; and from A draw AN perpendicular to CB: show that BT is to BC as BN is to NA.

(Fig. 320, Plate XXII.)—Draw CT, AB intersecting at R.

Because TRB, TBC are right angles, $ABN = CTB$; hence triangles ABN, CTB are equiangular, $\therefore BT : BC :: BN : NA$ (VI. 4).

EXERCISE 321.

Find a straight line such that the perpendiculars on it from three given points shall be in a given ratio to each other.

(Fig. 321, Plate XXII.)—Let A, B, C be the three given points, and let $a : c$ be the ratio of the perpendicular from A to the perpendicular from C on the required line; and $c : b$ the ratio of the perpendicular from C to that from B.

Join A, C, and divide the line AC at D in the ratio of $a : c$ (VI. 10); join C, B, and divide the line CB at E in the ratio of $c : b$; through D, E draw the straight line $HDEG$; then $HDEG$ is the required straight line.

From A, B, C draw AH, BK, CG perpendicular to the line $HDEG$. Now, in the similar triangles AHD and CGD, $AH : CG :: AD : DC$ (VI. 4); but $AD : DC :: a : c$ (*Cons.*), $\therefore AH : CG :: a : c$. Again, in the similar triangles CEG and BEK, $CG : BK :: CE : EB :: c : b$ (*Cons.*). Again, since $AH : CG :: a : c$, and $CG : BK :: c : b$, $\therefore AH : BK :: a : b$. Therefore HG is the straight line required.

Note.—This may also be done by joining A, B, and B, C, and dividing the joining lines in the given ratios of $a : b$, and of $b : c$.

EXERCISE 322.

If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

(Fig. 322, Plate XXII.)—Let the perpendiculars from A, C meet the straight line BD at B, D respectively; join AC cutting BD at M.

Now M is the third fixed point; for since triangles ABM, CDM are equiangular, $CD : AB :: CM : AM$, hence the point M is fixed.

EXERCISE 323.

P and Q are fixed points; AB and CD are fixed parallel straight lines; any straight line is drawn from P to meet AB at M, and a straight line is drawn from Q parallel to PM meeting CD at N: show that the ratio of PM to QN is constant, and thence show that the straight line through M and N passes through a fixed point.

(Fig. 323, Plate XXII.)—Because PM is parallel to QN, and PQ meets them, the angle QPM = angle PQN **(I. 29)**; and since PM and AM are respectively parallel to QN and DN, the angle PMF = angle QNG (*Ex*, 27, PART I.); and ∴ the triangles MPF and QNG are **(I. 32)** equiangular and **(VI. 4)** similar, and $FP : PM :: GQ : QN$, and, alternately, $FP : GQ :: PM : QN$. But $FP : GQ$ is constant, ∴ $PM : QN$ is also constant. Again, from similar triangles, FPM and GQN, and FMS and GNS, $PF : FM :: QG : GN$; and $MF : FS :: NG : GS$, ∴ $PF : FS :: QG : GS$, and, alternately, $PF : QG :: FS : SG$. But $PF : QG$ is constant, ∴ $FS : SG$ is constant, and the point S, through which MN passes, is fixed.

EXERCISE 324.

Through a given point within a given angle draw a straight line meeting the legs of the angle so that it may be divided at the point in a given ratio.

(Fig. 324, Plate XXII.)—Let HAR be the angle, and K the point within the legs.

Draw KW parallel to AH, meeting AR at W; in WR take M so that $AW : WM$ in the given ratio; join MK and produce it to meet AH at S: SM is the line required.

$SK : KM :: AW : WM$ **(VI. 2)**; that is, in the given ratio.

EXERCISE 325.

Through a given point draw a straight line so that the parts of it intercepted between that point and the perpendiculars drawn to the straight line from two other given points may have a given ratio.

(Fig. 325, Plate XXII.)—Let A be the point through which the line is to be drawn, and B, C the other given points.

Join AB and produce it to E, so that $AE : AB$ may be in the given ratio.

Join CE, and draw AD perpendicular to it: AD is the line required.

Draw BM parallel to CE meeting AD at M. CD, BM are at right angles to AD, and $AD : AM :: AE : AB$, *i.e.*, in the given ratio.

EXERCISE 326.

Draw through a point a straight line, so that the part of it intercepted between a given line and a given circle may be divided at the given point in a given ratio.

(Fig. 326, Plate XXII.)—Let K be the given point and H the centre of the given circle.

Join KH and produce it both ways to meet the circle again at A, and the given line at S; cut KA at Q so that $SK : KQ$ may be in the given ratio, and draw QW parallel to the given line meeting the circumference at W; join WK and produce it to meet the given line at R: WR is the line required.

For $KR : KW :: SK : KQ$; that is, in the given ratio. If the given ratio be less than the ratio of SK to KA, or greater than that of SK to KN, the solution is impossible, AN being a diameter.

EXERCISE 327.

Given two circles which intersect, draw through either point of intersection a line cutting the circles, so that the chords intercepted may be in a given ratio?

(Fig. 327, Plate XXII.)—Cut AB, the line joining their centres in D, so that the segments may be in the required ratio (VI. 10), and join D with M, one of the points of intersection; through M draw NMH at right angles to DM, meeting the circles at N, H: NMH is the line required.

Draw AR, BQ perpendicular to MN, MH. Then $NM : MH :: AD : DB$ (III. 3) and (V. 15); that is, in the given ratio.

PLATE XXI.

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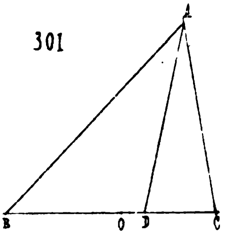

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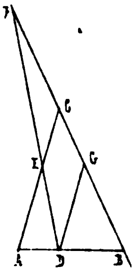

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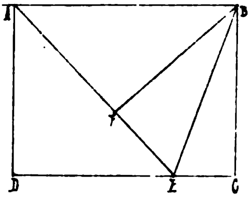

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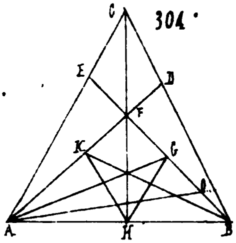

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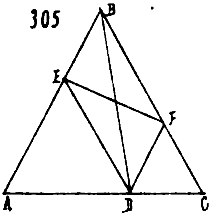

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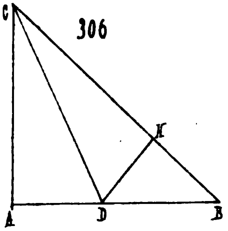

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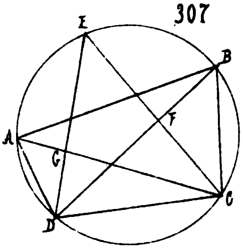

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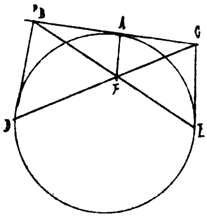

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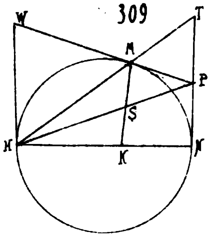

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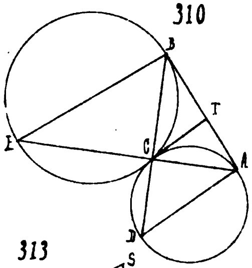

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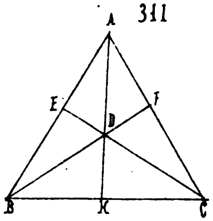

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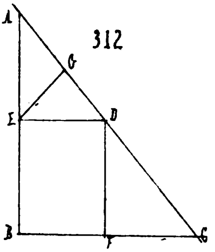

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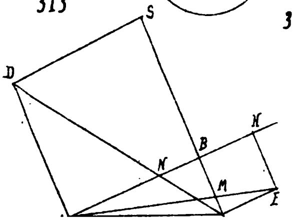

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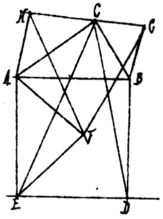

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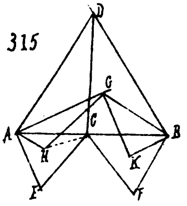

PLATE XXII.

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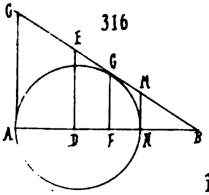

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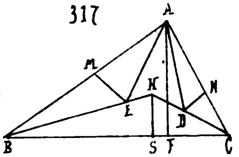

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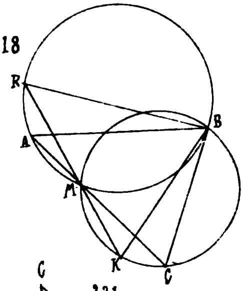

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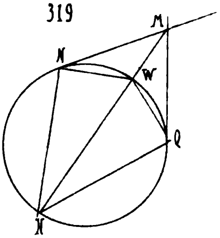

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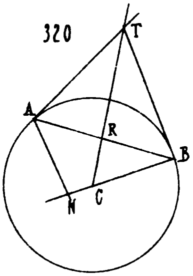

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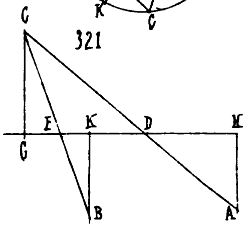

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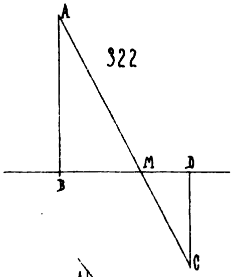

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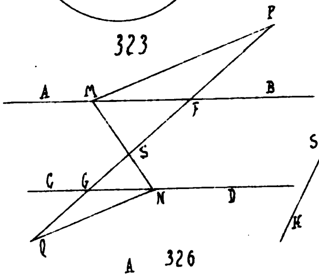

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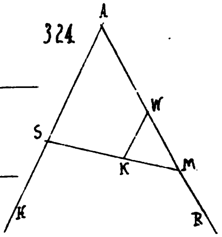

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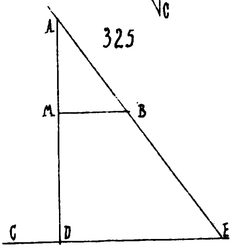

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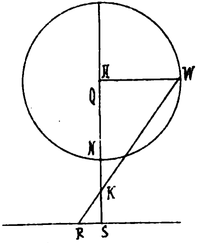

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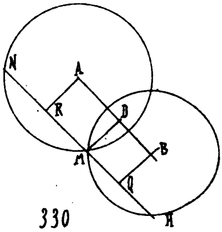

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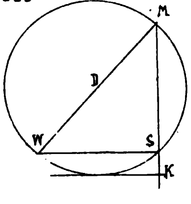

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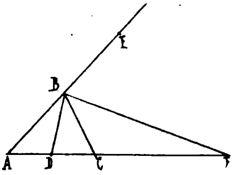

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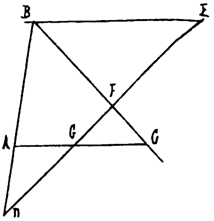


EXERCISE 328.

From what point in a circle must a tangent be drawn so that a perpendicular on it from a given point in the circumference may be cut by the circle in a given ratio.

(**Fig. 328, Plate XXII.**)—Let *M* be the given point in the circumference of the given circle *WMS*, and *D* the centre.

Draw the diameter *MDW* and produce it to *N*, making *MW* : *WN* in the given ratio; draw the tangent *NO*: then *O* is the point required.

Produce *NO*; draw *MK* perpendicular to it at *K*, and let *MK* cut the circle at *S*, and join *WS*.

The angle at *S* is right (III. 31), ∴ *WS* is parallel to *NK*, hence *MS* : *SK* :: *MW* : *WN*, *i.e.*, in the given ratio.

EXERCISE 329.

If the interior and exterior vertical angles of a triangle be bisected by two straight lines cutting the base and base produced, the line made up of the base and its produced part is cut harmonically.

(**Fig. 329, Plate XXII.**)—Let *ABC* be a triangle, and let the interior vertical angle *ABC* and the exterior vertical angle *CBE* be bisected by *BD* and *BF* cutting *AC* in *D*, and *AC* produced in *F*: then the line *AF* is cut harmonically in *D* and *C*.

Because the angle *ABC* is bisected by *BD*, *AB* : *BC* :: *AD* : *DC* (VI. 3); and because the angle *CBE* is bisected by *BF*, *AB* : *BC* :: *AF* : *FC* (VI. *Prop. A*); and ∴ from equality of ratios, *AF* : *FC* :: *AD* : *DC*; hence *AF* is cut harmonically.

EXERCISE 330.

If through G, the middle point of the base AC of a triangle ABC a line DGFE be drawn meeting BA produced in D, BC in F, and BE parallel to the base at E: DGFE is cut harmonically.

(**Fig. 330, Plate XXII.**)—Because the parallel *AC* is bisected at *G*, and the triangles *FGC*, *FEB* are equiangular, ∴ *ED*:*DG*::*EB*:*GC* :: *EF* : *FG* (VI. 2 and 4), hence *DGFE* is cut harmonically.

EXERCISE 331.

If through any point in the produced diameter of a circle two straight lines be drawn making equal angles with the diameter and cutting the circumference in four points (**a**), *the straight lines joining the opposite points intersect each other in the diameter; and* (**b**) *their point of intersection and the convex circumference divide the produced diameter harmonically.*

(**Fig. 331, Plate XXIII.**)—Let C be any point in the diameter AB produced, and let the straight lines CEF and CGH make equal angles with AC and cut the circumference in E, F, G, H; then the straight lines EH and FG, joining the nearer point of one with the more remote point of the other, cut AB in the same point D; and AC is divided harmonically in D and B.

Join AF, AH, AE, EB.

(**a**) Because the angle $ACF =$ angle ACH (*hyp.*), $CE = CG$ (III. 8); and because $FC \cdot CE = HC \cdot CG$ (III. 36), and that $CE = CG$, FC must be equal to HC (I. 36), and $EF = HG$ (*Ax.* 3); and $AF = AH$ (I. 4), and $\therefore$ the angle $FEH =$ angle FGH (III. 28 and 27), and their supplements DEC and DGC are equal (*Ax.* 3), and hence EH and GF cut AC in the same point D.

(**b**) Because $AF = AH$, the angle $AEF =$ angle AEH (III. 28 and 27); but as the angle AEB is a right angle (III. 31), the angles FEA and BEC are together equal to a right angle (I. 13), and are therefore equal to $HEB + AEH$; but $AEF = AEH$, $\therefore$ $HEB = BEC$. Now, since the interior angle DEC and the exterior angle HEF, of the triangle DEC, are bisected by EB and EA, $AC:CB::AD:DB$ (*Ex.* 329), $\therefore$ AC is cut harmonically in D and B.

EXERCISE 332.

If a straight line be drawn from the point of intersection of two tangents to a circle to any point in the concave circumference, it will be divided harmonically by the convex circumference and the chord joining the points of contact.

(**Fig. 332, Plate XXIII.**)—Let P be the point of intersection of two tangents AP, BP to the circle $ADBC$; let $PDEC$ be any secant drawn from P; and let A, B be joined; **then the line PC is cut harmonically in D and E.**

The rectangle $CP\cdot PD = AP^2$ (III. 36) $= PE^2 + AE\cdot EB$ (*Ex.* 253, Part I.) $= PE^2 + CE\cdot ED$ (III. 35). But $PE^2 = EP\cdot PD + EP\cdot ED$ (II. 2); and $\therefore$ $CE\cdot ED + EP\cdot PD + EP\cdot ED = CP\cdot PD = CE\cdot PD + EP\cdot PD$ (II. 1). From these equals take $EP\cdot PD$, and (*Ax.* 3) $CE\cdot ED + EP\cdot ED = CE\cdot PD$. But $CE\cdot ED + EP\cdot ED = (CE + EP)\cdot ED = CP\cdot ED$; and $\therefore$ (*Ax.* 1) $CP\cdot ED = CE\cdot PD$; and hence (VI. 14) $CP : PD :: CE : ED$, and therefore CP is cut harmonically in D and E.

EXERCISE 333.

If two tangents BA, BC be drawn to a circle, and a third, $DHEF$, touching that part of the arc between the former at H, and cutting BA, BE, and AC produced at D, E, and F respectively: then $DHEF$ is cut harmonically.

(Fig. 333, Plate XXIII.)—Draw EG parallel to BA meeting AC at G.

Angles ECG, EGC are equal, $\therefore$ $EG = EC = HE$ and $DF : FE :: AD = DH : EG = HE$, hence $DHEF$ is cut harmonically at H, E.

EXERCISE 334.

If two circles intersect at right angles, then any line through the centre of either is cut harmonically by the other circumference.

(Fig. 334, Plate XXIII.)—Let $CNQA$ be drawn through the centre M of one of the circles BQC, CQ being its diameter, and let it intersect the other at N, A: then the line $CNQA$ is divided harmonically by the circumferences.

Let B be one of the points of intersection, and join AB, QB, NB, MB, CB.

Produce AB to W.

Because angle MBN = angle A, and $MBQ = MQB$ = angle $A + ABQ$, $\therefore$ $ABQ = NBQ$, hence $AQ : QN :: AB : BN$ (VI. 3), and because QBC is a right angle (III. 31), BC bisects the exterior angle NBW, $\therefore$ $AC : CN :: AB : BN$, consequently $AQ : QN :: AC : CN$, $\therefore$ $CNQA$ is divided harmonically.

EXERCISE 335.

If through the extremities and points of section of a straight line divided harmonically, straight lines be drawn to any point without *the line, these four lines divide harmonically any straight line that falls upon them, or their continuations.*

(Fig. 335, Plate XXIII.)—Let the straight line *AB* be divided harmonically in *D* and *E*, and let *C* be any point without it; join *CA*, *CD*, *CE*, *CB*; then any straight line *FGHM*, falling on these lines, is divided harmonically in *G* and *H*.

Through *E* draw *WK* terminating in *BC* and *CD* produced, so that *WE* may be equal to *EK* (*Ex.* 152, Part I.), and through *H* draw *SN* parallel to *WK*.

Because *SN* is parallel to *WK*, and that *CE* bisects *WK*, it also bisects *SN* (*Ex.* 156, Part I.). Again, since $WE = EK$, *FC* is parallel to *WK* (*Ex.* 330, *Cor.*, Part II.), and hence also to *SN*. Now, because *FM* is drawn through the middle point of *SN*, the base of the triangle *SCN*, it is divided harmonically in *G* and *H* (*Ex.* 336, Part II.).

EXERCISE 336.

To cut a given straight line harmonically.

(Fig. 336, Plate XXIII.)—Let *AB* be the given straight line.

Through the point *A* draw any line *CD* making *AC* equal to *AD*; join *DB*; from *C* draw *CE* to any point in *BD*; from *E* draw *EF* parallel to *CD*, and cutting *AB* in *F*: then *AB* is cut harmonically in *G* and *F*.

Because, in the triangle *ABD*, *FE* is parallel to *AD*, $AB : FB :: AD : FE$ (VI. 2); that is, $AB : FB :: AC : FE$, *AD* being = to *AC* (*Cons.*). Again, because the triangles *ACG* and *EFG* are similar, $AC : AG :: EF : FG$ (VI. 4), and alternately $AC : FE :: AG : FG$; but it has been shown that $AC : FE :: AB : FB$, and ∴ by equality of ratios $AB : FB :: AG : FG$.

EXERCISE 337.

AB is a given straight line, and D a given point in it; it is required to find a point P in AB produced, such that AP is to PB as AD is to DB.

(Fig. 337, Plate XXIII.)—On AB describe any segment of a circle, and complete the circle of which it is the segment (III. 25). Let AFB be the segment described on AB. Bisect the arc of the other segment AEB in E (III. 30), and through D draw the line EDF; join AF, FB; produce AF to G; bisect the angle BFG by the line FP meeting AB produced in P: then P is the required point.

Because the angle AFB is bisected by the line DF (*Cons.*, and III. 27), and the external angle BFG is bisected by FP (VI. 3 and A), $AD : DB :: AF : FB$, and $AP : PB :: AF : FB$; and $\therefore$ (V. 11) $AP : PB :: AD : DB$.

EXERCISE 338.

ANOTHER METHOD.

(Fig. 338, Plate XXIII.)—Through D draw any line CE making DC equal to DE; join AE and produce it to meet CB, joined and produced in F; through F draw FP parallel to CE, and meeting AB produced in P; then P is the required point.

Because FP is parallel to ED, and $ED = DC$, $FP : ED :: PA : AD$; that is, $FP : DC :: PA : AD$. Again, since the triangles FPB and DCB are similar, $FP : DC :: PB : BD$; and $\therefore$ (V. 11) $PA : AD :: PB : BD$, or, alternately, $PA : PB :: AD : DB$.

EXERCISE 339.

A, B, C are points in order in a straight line: find a point P in the straight line so that PB may be a mean proportional between PA, PC.

(Fig. 339, Plate XXIII.)—On AC describe a semicircle; draw BM meeting the arc at M, and making ABM equal to half a right angle; draw MP perpendicular to AC meeting it at P: P is the point required.

The angles at M, B are equal, $\therefore PB = PM$, and $PM^2 = PA \cdot PC$, $\therefore PB^2 = PA \cdot PC$, and is $\therefore$ a mean proportional between PA, PC, hence P is the point required.

EXERCISE 340.

Draw a parallel to one of the sides AB of a triangle ABC, so that it may be a mean proportional between the segments into which it divides one of the other sides AC.

(Fig. 340, Plate XXIII.)—Produce BA to R making $AR = BA$, and join CR; draw AK meeting CR at K and making angle $CAK = CBA$; draw KNM parallel to BA, meeting CA, CB in N, M respectively: NM is the line required.

$CMK = CAK$, each being equal to angle B, $\therefore$ the points A, M, C, K are in the circumference of a circle; $\therefore$ $CN \cdot NA = KN \cdot NM$ (III. 35), but since CA bisects BR, it also bisects MK (*Ex.* 156, Part I.), $\therefore$ $CN \cdot NA = NM^2$, hence NM is a mean proportional between CN, NA.

EXERCISE 341.

ANOTHER METHOD.

(Fig. 341, Plate XXIII.)

Let CK, drawn parallel to AB, be a third proportional to AB, AC; join AK cutting BC in M, and draw MN parallel to AB cutting AC in N: then MN is a mean proportional between NA, NC.

$AB : AC :: MN : NC$ (VI. 4), and $AC : CK :: AN : MN$, but $AB : AC :: AC : CK$, $\therefore$ $AN : MN :: MN : NC$, hence MN is a mean proportional between AN, NC.

EXERCISE 342.

If from the extremities of one of the equal sides of an isosceles triangle inscribed in a circle tangents be drawn to the circle, and produced to meet: two lines drawn to any point in the circumference from the vertex and the point in which the tangents meet will divide the base, produced if necessary, in continued proportion.

(Fig. 342, Plate XXIII.)—Let ABC be an isosceles triangle inscribed in the circle, AB being equal to BC; let AT, BT be tangents from A, B; let TD, BD be two lines drawn from T and B to any point D in the arc ADC cutting the base AC of the triangle ABC in M and N: then AC is cut in continual proportion in M and N.

Join AD.

The angle TBA = the angle BCA (III. 32) = angle BAC (I. 5), and $\therefore$ TB is parallel to AC (I. 27). But the angle TAB = angle TBA (since $TB = TA$, III. 36); hence the triangles ABC and ATB are similar, and $BA : AC :: TB : BA$. Again, in the isosceles triangle ABC, $AB^2 = BN^2 + AN \cdot NC$

(*Ex.* 253, Part I.) $= BN^2 + DN \cdot NB = DB \cdot BN$ (**II. 3**); and $\therefore BN : AB :: AB : BD$; and, by multiplication, or *Ex. æquo*, $BN : AC :: TB : BD$. But, because MN is parallel to TB, $TB : BD :: MN : ND$ (**VI. 2**); and since AND and BNC are similar, $AN : ND :: BN : NC$, $\therefore AN : MN :: AC : NC$; and, by conversion, $AN : AN - MN :: AC : AC - NC$, or $AN : AM :: AC : AN$; that is, $AC : AN :: AN : AM$.

EXERCISE 343.

AD and AE bisect the interior and exterior angles at A of a triangle ABC, and meet the base at D and E; and O is the middle point of BC: show that OB is a mean proportional between OD and OE.

(Fig. 343, Plate XXIII.).

Because OB equals half the sum of BD, DC, and OD half their difference, and OE equals half the sum of BE, EC and OB half their difference, $\therefore$ (**VI. 3** and *A*) and (**V. 15**), $OE : OB :: OB : OD$, hence OB is a mean proportional between OD and OE.

EXERCISE 344.

ABCD is a parallelogram; from B a straight line is drawn cutting the diagonal AC at F, the side DC at G, and the side AD produced at E: show that the rectangle EF, FG is equal to the square on BF.

(Fig. 344, Plate XXIII.)—Triangles AFE, BFC are equiangular, as also are AFB, CFG, $\therefore EF : BF :: AF : CF :: BF : FG$ (**VI. 4**), hence $EF \cdot FG = BF^2$.

EXERCISE 345.

AB is a diameter, and P any point in the circumference of a circle; AP, BP are joined and produced if necessary; from any point C in AB a straight line is drawn at right angles to AB meeting AP at D and BP at E, and the circumference at F: show that CD is a third proportional to CE and CF.

(Fig. 345, Plate XXIII.)—$CE \cdot CD = AC \cdot CB$ (*Ex.* 148, *Cor.*) $= CF^2$, $\therefore$, hence CD is a third proportional to CE, CF.

Otherwise:—Because the triangles ADC and ECB are similar, $AC : CD :: EC : CB$; and $\therefore AC \cdot CB = CD \cdot EC$; but $AC \cdot CB = CF^2$, hence $CD \cdot EC = CF^2$; and $\therefore EC : CF :: CF : CD$, and CD is a third proportional to EC, CF.

EXERCISE 346.

If the extremity of a chord of one circle be the centre of another, the circumference of the latter circle and the common chord cut the former chord in continual proportion.

(Fig. 346, Plate XXIV.)—Let the extremity C, of the chord AC, of the circle ABD, be the centre of the circle DEB; then AC is cut in continual proportion by BD in F, and the circumference of the circle DEB in E.

Join CB.

Because C is the centre of the arc BCD, it is evident from II. 3 and *Ex.* 156, Part I., that the rectangle $AC \cdot CF = CB^2 = CE^2$, and $\therefore AC : CE :: CE : CF$.

EXERCISE 347.

From the obtuse angle of a triangle draw a line to the base which shall be a mean proportional between the segments of the base.

(Fig. 347, Plate XXIV.)—Circumscribe the triangle ABC by a circle (IV. 5), and produce BA to D making $AD = AB$; draw DE parallel to AC meeting the circle at E, and join BE cutting AC in F: BF is the line required.

Because BD is bisected in A, and AF is parallel to DE, BE is bisected at F, $\therefore CF \cdot FA = BF^2$ (III. 35), hence BF is a mean proportional between the segments of the base.

For another method of solution see *Ex.* 246.

Note.—If DE produced meets the circumference again at R, and BR be drawn cutting AC in H, then BH is evidently another line drawn in the manner required; and since E and R are the only points in the circumference to which lines drawn from the vertex are bisected by the base, the following theorem is suggested:—In an obtuse-angled triangle there cannot be more than two lines drawn from the vertex to the base, which shall be mean proportionals between the segments into which they divide the base.

EXERCISE 348.

If ABC be a triangle inscribed in a circle, and BD be drawn parallel to the tangent at A to meet AC, or AC produced in D, show that AB is a mean proportional between AC and AD.

(Fig. 348, Plate XXIV.)—Because EA is a tangent, the angle EAB = angle ACB (III. 32); and because EA is parallel to BD, the angle EAB = angle ABD (**I. 29**); and ∴ the angle ACB = angle ABD (Ax. 1), and as the angle BAC is common, the triangles ABC and ABD are similar, and $CA : AB :: AB : AD$ (**VI. 4**), therefore AB is a mean proportional between AC and AD.

OTHERWISE:—$DBA = EAB$ (**I. 29**) $= BCA$ (**III. 32**), ∴ AB is a tangent to the circle circumscribing DBC, hence $CA \cdot AD = AB^2$ (**III. 36**), ∴ AB is a mean proportional between AC, AD.

EXERCISE 349.

AB is any chord of a circle; AC, BC are drawn to any point in the circumference, and cut the diameter perpendicular to AB at D, E; if O be the centre of the circle, show that $OD \cdot OE$ is equal to the square on the radius.

(Fig. 349, Plate XXIV.)—Join OA and OC. Because $OA = OC$, the angle OAC = angle OCA; and as the three angles OAC, OCA, AOC are together equal to two right angles, $2OCA + AOC$ = two right angles. But AOC = twice ABC (**III. 20**); and ∴ $2OCA + 2ABC$ = two right angles, and $OCA + ABC$ = one right angle.

Again, $BEH + ABC$ = one right angle; and ∴ $OCA = BEH$. But the angle EOC is common to the two triangles OCD and OCE, and, as OEC has been shown to be equal to OCD, these triangles are similar, ∴ $EO : OC :: OC : OD$, and hence $EO \cdot OD = OC^2$.

EXERCISE 350.

Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in a given ratio.

(Fig. 350, Plate XXIV.)—Construct a triangle HWX, having the angle at W equal to that in the given segment ABC, and having $WH : WX$ in the given ratio.

Join AC; draw AB meeting the arc at B, and making the angle at A equal to that at H: then the arc is divided at B in the manner required.

Join CB.

Triangles ABC, HWX are equiangular, ∴ $AB : BC :: HW : WX$, *i.e.*, in the given ratio.

OTHERWISE:—Complete the circle and bisect the arc AMC in M; cut AC in D so that $AD : DC$ is in the given ratio;

join MD and produce it to cut the given arc in B : then ABC is divided at B in the manner required. For since BD bisects the angle B (**III. 27**), $\therefore AB : BC :: AD : DC$ (**VI. 3**)—that is, in the given ratio.

EXERCISE 351.

Given the vertical angle, the ratio of the segments of the base by a perpendicular from the vertex, and the perpendicular, to construct the triangle.

(**Fig. 351, Plate XXIV.**)—Take any line AB and on it describe a segment of a circle containing an angle equal to the given vertical angle; divide AB in D in the given ratio; from D draw DC at right angles to AB; join AC, BC; from CD cut off CH equal to the given perpendicular, and through H draw EHF parallel to AB : then ECF is the triangle required. Because EF is parallel to AB, $EH : HF :: AD : DB$, by equality of ratios; and $\therefore$ ECF is the required triangle, having the vertical angle ratio of segments of base, and perpendicular equal to those given.

EXERCISE 352.

Construct a triangle having given one side, the opposite angle, and the ratio of the remaining sides.

(**Fig. 352, Plate XXIV.**)—On the given side describe a segment of a circle containing an angle equal to the given angle (**III. 33**), and divide the arc in C so that its chords may be in the given ratio (*Ex.* 350). Join AC, CB : then ABC is evidently the triangle required.

EXERCISE 353.

On a given straight line AB as hypotenuse describe a right-angled triangle which shall have its sides in continued proportion.

(**Fig. 353, Plate XXIV.**)—On AB describe a semicircle and cut it in medial section at D; draw DC at right angles to AB meeting the arc at C, and join CA, CB : then ABC is the triangle required.

$BC^2 = AB \cdot BD = AD^2$, $\therefore BC = AD$, and $AC^2 = AB \cdot AD = AB \cdot BC$, $\therefore AB : AC :: AC : BC$.

EXERCISE 354.

Given the vertical angle, the line bisecting the base, and the angle which the bisecting line makes with the base: to construct the triangle.

(Fig. 354, Plate XXIV.)—Take any line DE and on it describe a segment of a circle containing an angle equal to the given vertical angle; bisect DE in G, and at G make the angle EGF or DGF equal to the angle which the bisecting line makes with the base; produce GF if necessary till GC be equal to the line bisecting the base; produce DE both ways if necessary, and through C draw CA and CB parallel to DF and EF; then ACB is the triangle required.

Because AC and DF are parallel, $AG : GC :: DG : GF$ (**VI. 2**); and, for the same reason, $BG : GC :: EG : GF :: DG : GF$; and $\therefore AG = GB$. Therefore ACB is the required triangle; for it is evident (*Ex.* 27, PART I.) that the angle ACB is equal to the angle DFE.

Note.—If GF be greater than the given bisecting line, from it cut off a part equal to that line, and through its extremity draw two lines parallel to DF and EF respectively; these two lines will form, with that part of DE which they include, the required triangle: the proof is the same as that given already.

EXERCISE 355.

Given the base, the altitude, and the ratio of the sides of a triangle: to construct it.

(Fig. 355 Plate XXIV).—Let AB be the given base, p the altitude, and a to b in the ratio of the sides; it is required to construct the triangle.

Divide AB in C so that $AC : CB :: a : b$ (**VI. 10**); produce AB, and cut it in D so that $AD : DB :: a : b$. Then on CD as diameter describe a circle; draw GH parallel to AB and at a distance from it equal to p, cutting the arc in E; join AE, BE: then ABE is the triangle required.

Join EC, ED, and EK, K being the centre of the circle.

Then, because (*Cons.*) $AC : CB :: a : b$, and $AD : DB :: a : b$, $\therefore AC : CB :: AD : DB$, by equality of ratios; and, alternately, $AC : AD :: CB : DB$, and inversely, $AD : AC :: DB : BC$; and, by composition and division, $AD + AC : AD - AC :: DB + BC : DB - BC$. But $AD + AC = 2AC + 2CK = 2AK$; $AD - AC = CD = 2CK$; $DB + BC = CD = 2CK$; and $DB - BC = CK + BK - BC = CB + 2BK - CB = 2BK$; and $\therefore 2AK : 2CK :: 2CK : 2BK$, or $AK : CK :: CK : BK$, and hence (**VI. 16**) $AK \cdot BK = CK^2$, and $\therefore$ the angle AEB is bisected by the line EC (*Ex.* 289), and hence (**VI. 3**) $AE : EB :: AC : CB :: a : b$, and $\therefore$ ABE is the triangle required.

EXERCISE 356.

Given in any triangle the base, the ratio of the sides, and the distance between the points in which the internal and external bisectors cut the base : to construct the triangle.

(**Fig. 356, Plate XXIV.**)—Let AB be the given base, and divide it in C in the given ratio of the sides; produce CB, and make CD equal to the given distance between the bisectors of the internal and external angles.

On CD describe a circle; take any point in the circumference E and join it with A, B; then AEB is the triangle required.

Join K, the centre of the circle, or middle of CD, with E.

CK or EK is a mean proportional between AK and KB (*Ex.* 290), and hence $AK : KE :: KE : KB$, and (**VI. 6**) the triangles AKE and EKB are similar, and the angle $KEB =$ angle KAE. But the angle $KCE = CAE + AEC$ (**I. 32**); and as the angle $KCE = KEC$ (**I. 5**), $KEC = CAE + AEC$, and since $KEB = CAE$, the remaining angles AEC and BEC are equal, and $\therefore AE : EB :: AC : CB$ (**VI. 3**).

Again, join DE and produce AE to F; and since $AD : DB :: AC : CB :: AE : EB$, the angle BEF is also bisected by DE; and $\therefore AEB$ is (**VI.** A) the triangle required.

EXERCISE 357.

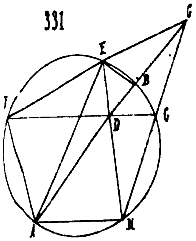
Given the vertical angle, the base, and a line drawn from either of the angles at the base to cut the opposite side in a given ratio : to construct the triangle.

(**Fig. 357, Plate XXIV.**)—Let AB be equal to the given base; divide it at D in the given ratio (**VI. 10**); on AD describe a segment of a circle containing an angle equal to the given angle; from B draw BC equal to the given line, and, if necessary, meeting the circumference again at C; join AC, DC, and through B draw BE parallel to DC meeting AC produced in E; then ABE is the required triangle.

Because DC is parallel to BE, $AC : CE :: AD : DB$ (**VI. 2**), AE is cut in the given ratio; and AB is equal to the given base, and BC equal to the given line from the base angle B cutting AE as required.

PLATE XXIII.

331



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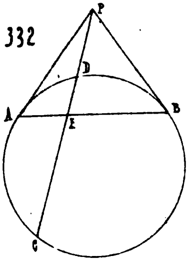

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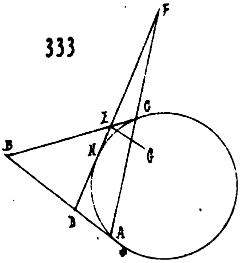

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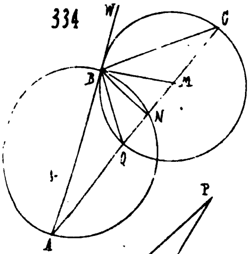

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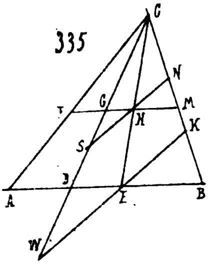

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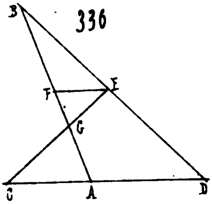

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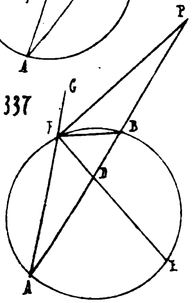

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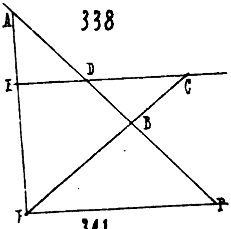

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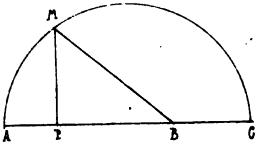

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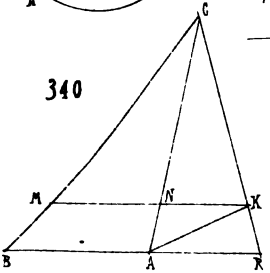

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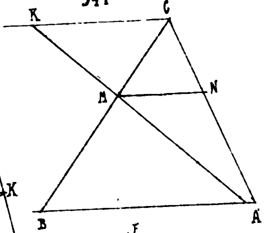

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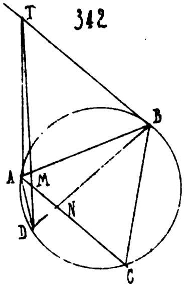

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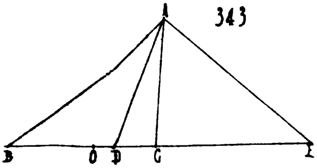

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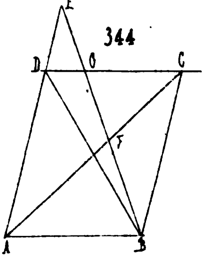

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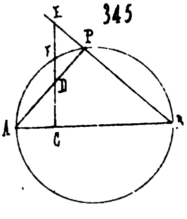

PLATE XXIV.

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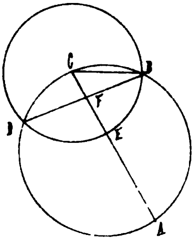

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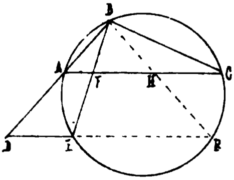

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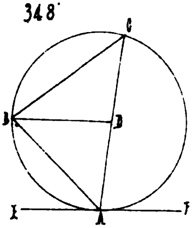

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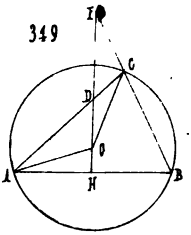

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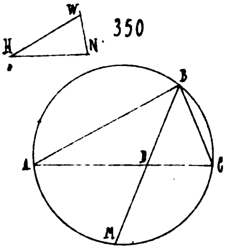

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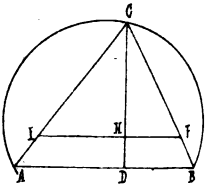

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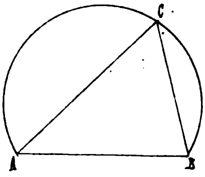

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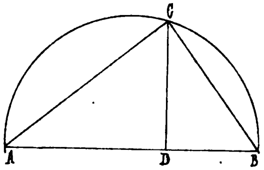

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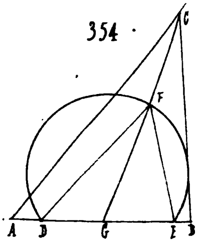

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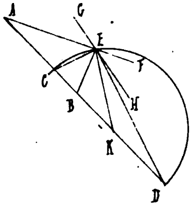

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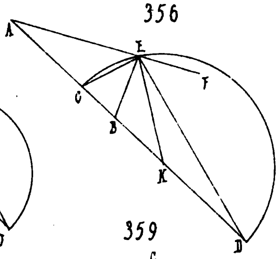

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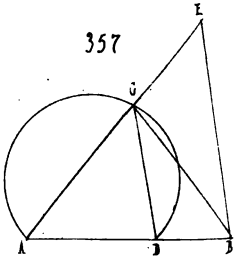

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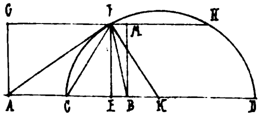

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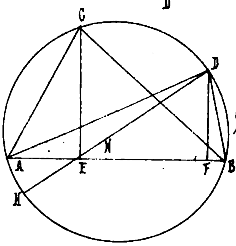

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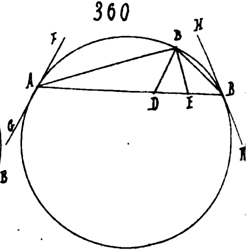


EXERCISE 358.

Given the base, the area, and the ratio of the sides of a triangle: to construct it.

(Fig. 358, Plate XXIV.)—Let *AB* be the given base; divide it in *C* in the ratio of the sides; produce *AB*, and divide it in *D* so that *AD* is to *DB* in the ratio of the sides, or as *AC* to *CB*; on *CD* describe a semicircle; to *AB* apply a right-angled parallelogram *AGMB* double of the given area, and let its side *GM* cut the arc in *F*; join *AF*, *FB*; then *AFB* is the triangle required.

Join *CF*. Then, since *AC* : *CB* :: *AD* : *DB*, and that a semicircle is described on *CD*, and *AF* and *FB* joined, *F* being any point in the arc, *AF* : *FB* :: *AC* : *CB* (*Ex.* 290, Part II.); but *AC* : *CB* in the ratio of the sides (*Cons.* **VI. 10**), ∴ *AF* : *FB* in the ratio of the sides. Again, the triangle *AFB* is half of the parallelogram *AGMB* (**I. 41**); but *AGMB* is double of the given area, ∴ the triangle *AFB* is equal to the given area, and is hence the triangle required.

Note.—If *GM* or *GM* produced do not meet the arc the problem is impossible; if it cut it in two points, as in *F* and *H*, there will be two entirely different triangles, each of which will answer the conditions of the problem.

EXERCISE 359.

Triangles on the same base, and with equal vertical angles, are to one another as the products of their sides.

(Fig. 359, Plate XXIV.)—Draw the diameter *DH*.

Since the triangles are on the same base, and have equal vertical angles, they are in the same segment of a circle (**III. 21**). Let *ABC* and *ABD* be the triangles. From their vertices *C*, *D*, let fall the perpendiculars *CE*, *DF*.

The triangle *ACB* : triangle *ADB* :: *CE* : *DF* (**VI. 1**, *Cor.* 2); and the rectangle $AC \cdot CB = HD \cdot DE$, and $AD \cdot DB = HD \cdot DF$ (**VI.** *C*); ∴ $AC \cdot CB : AD \cdot DB :: HD \cdot CE : HD \cdot DF :: CE : DF$. But it was shown that the triangle *ACB* : *ADB* :: *CE* : *DF*, and, ∴, by equality of ratios, $ACB : ADB :: AC \cdot CB : AD \cdot DB$.

EXERCISE 360.

ABC is a triangte inscribed in a circle, BD and BE are lines drawn to the base AB parallel to the tangents at A, C: show that BD is equal to BE, and that $AE : CD :: AB^2 : BC^2$.

(**Fig. 360, Plate XXIV.**)—Because FG is a tangent and AB a chord from the point of contact A, the angle FAB = angle ACB (**III. 32**); and since FG is (*hyp.*) parallel to BD, the angle FAB = angle ABD (**I. 29**), $\therefore$ the angle ABD = angle ACB (*Ax.* 1). Again, since HK is a tangent at C, and BC a chord, the angle BCH = angle BAC; but, since HK is parallel to BE, the angle BCH = angle CBE, $\therefore$ the angle CBE = angle BAC. Now, since the angles ABD and BAD are together equal to the angles BCE and CBE, and that the angle $BDE = BAD + ABD$, and $DEB = BCE + EBC$, the angle BDE = angle DEB (*Ax.* 1), and hence $BD = BE$.

Again, because the angles BDE and DEB are equal, their supplements ADB and BEC are equal; but the angle $ADB = DCB + DBC$, and $BEC = BAE + ABE$, and $\therefore$ the triangles ABE and CBD are similar; and, since they have the same altitude, $ABE : CBD :: AE : DC$ (**VI. 1**); but (**VI. 19**) $ABE : CBD :: AB^2 : BC^2$, and, $\therefore$, by equality of ratios, $AE : DC : AB^2 : BC^2$.

EXERCISE 361.

Perpendiculars are let fall from two opposite angles of a rectangle on a diagonal : show that they will divide the diagonal into equal parts if the square on one side of the rectangle be double that on the other.

(**Fig. 361, Plate XXV.**)—Let the perpendiculars AE, CF be drawn from the opposite angles A, C of the rectangle $ABCD$ on the diagonal DB : then DE or BF is one-third of DB.

First Method—Because the triangles ABD and EAD are similar, $ABD : EAD :: DB^2 : AD^2$ (**VI. 19**); but $BD^2 = 3AD^2$ (*hyp.*), $\therefore$ (**VI. 1**) the triangle $ABD = 3ADE$, and hence $BD = 3DE$.

For a similar reason BF is one-third of DB, $\therefore$ EF is also one-third.

Second Method—Since the triangles ABD and AED are similar, and that $BD^2 = 3AD^2$, $AD^2 = 3DE^2$; and $\therefore$ $BD^2 = 9DE^2$, hence $BD = 3DE$.

EXERCISE 362.

If one side of a right-angled triangle be double of the other : show that the perpendicular from vertex on hypotenuse cuts it in the ratio of 1 : 4.

(Fig. 362, Plate XXV.)—In the right-angled triangle ABC, let BA be double of BC, and let BD be the perpendicular on the hypotenuse.

Because triangles CDB, ADB, are of the same altitude and similar, $DC : DA :: BC^2 : BA^2 :: 1 : 4$ (VI. 1 and 19).

OTHERWISE—Because ABC, ADB, BDC are similar (VI. 8), $DA = 2BD$, and $BD = 2DC$, $\therefore DC : DA :: 1 : 4$.

Or, $CA \cdot AD = AB^2 = 4CB^2 = 4CA \cdot CD$, $\therefore AD = 4CD$.

EXERCISE 363.

If an isosceles triangle be inscribed in a circle, having each of the sides double of the base, show that the square upon the radius is to the square upon one of the sides of the triangle as 4 to 15.

(Fig. 363, Plate XXV.)—From the centre K draw KF perpendicular to AC, and $CF = FA$ (III. 3). $AC^2 = CD^2 + AD^2$; but $AC^2 = 4AB^2$ (*hyp.*) $= 16AD^2$; and $\therefore CD^2 + AD^2 = 16AD^2$, and hence $CD^2 = 15AD^2$; that is, $AC^2 : CD^2 :: 16AD^2 : 15AD^2 :: 16 : 15$.

Because the triangles ACD and KCF are similar, $ACD : KCF :: CD^2 : CF^2 :: 15AD^2 : 4AD^2 :: 15 : 4$.

EXERCISE 364.

Construct an isosceles triangle equal to a given scalene triangle having the same vertical angle.

(Fig. 364, Plate XXV.)—Produce BC, the shorter side of the given triangle ABC, to E, so that BE may be a mean proportional between BA, BC; on BA take $BD = BE$, and join DE: DBE is the triangle required (VI. 15).

EXERCISE 365.

The straight line bisecting the angle B of a triangle ABC meets the straight lines drawn through A and C, parallel to BC and AB respectively, at E and F: show that the triangles CBE, ABF are equal.

(Fig. 365, Plate XXV.)—Triangles BAE, BCF are equiangular, $\therefore EB : BF :: BA : BC$ (VI. 4), hence triangles CBE, ABF are equal (VI. 15).

EXERCISE 366.

To divide a triangle into two parts in a given ratio by a straight line drawn through a given point in one of its sides.

(Fig. 366, Plate XXV.)—Let ABC be the given triangle, and P the given point; and let the given ratio be that of a to b.

Divide BC in the given ratio (**VI. 10**), so that CD is to DB as a to b; join AP, and draw DE parallel to it; join AD and EP.

The triangle AED = triangle EDP (**I. 37**); to each add the triangle EBD, and the triangle ABD = the triangle EBP (*Ax.* 1). Now, take each separately from the triangle ABC, and the triangle ACD = the quadrilateral $AEPC$ (*Ax.* 3); but the triangle ADC : triangle ADB :: CD : DB (**VI. 1**) :: a : b; and ∴ the quadrilateral $AEPC$, which is equal to ADC, : ADB :: a : b; that is, $AEPC$: EPB :: a : b. Hence ABC is divided by EP in the given ratio.

EXERCISE 367.

Find a point in a side of a triangle, from which two straight lines drawn, one to the opposite angle, and the other parallel to the base, shall cut off towards the vertex and towards the base, equal triangles.

(Fig. 367, Plate XXV.)—Cut the side BC of the given triangle ABC in D as in (**II. 11**); let DC the smaller segment be nearer to the base: D is the point required.

Join AD, and draw DE parallel to AC meeting AB at E.

CA : DE :: CB : BD (**VI. 4**), but CB : BD :: BD : DC (*Cons.*), ∴ CA : DE :: BD : DC, and the angles ACD, EDB are equal, hence triangles ACD, EDB are equal (**VI. 15**).

EXERCISE 368.

To bisect a triangle by a straight line parallel to one of its sides.

(Fig. 368, Plate XXV.)—Let ABC be any plane triangle: it is required to bisect it by a straight line parallel to one of its sides AC.

First Method:—Bisect AB in D, and find a mean proportional BE between AB and BD (**VI. 13**), and through E draw EF parallel to AC: then EF bisects the triangle ABC.

Because (*Cons.*) AB : BE :: BE : BD, and because AB :

BD :: the triangle ABC : triangle EBF (VI. 19, *Cor.*), and since AB is double of BD, the triangle ABC is double of the triangle EBF; that is, EF bisects the triangle ABC.

SECOND METHOD:—Divide AB in E so that AB^2 may be equal to twice BE^2; and draw EF parallel to AC; then EF bisects the triangle ABC.

The triangle ABC : the triangle EBF :: $AB^2 : EB^2$ (VI. 19); but AB^2 = twice EB^2, ∴ the triangle ABC = twice the triangle EBF.

EXERCISE 369.

To bisect a triangle by a straight line perpendicular to one of its sides.

(**Fig. 369, Plate XXV.**)—Let ABC be the given triangle. From B draw BW at right angles to AB, meeting AC produced if necessary in W; bisect AC in S, and on AB cut off AM a mean proportional between WA and AS; from M let fall the perpendicular MK; then MK bisects the triangle ABC. The triangles ABW and AKM are similar, having each a right angle and the common angle at A, and hence the triangle ABW : the triangle AKM :: $AW^2 : AM^2$ :: AW^2 (VI. 19) : $AW \cdot AS$ :: $AW : AS$. Again, the triangle ABW : the triangle ABC :: $AW : AC$ :: $AW : 2AS$ (VI. 1 and *Cons.*); and ∴ the triangle AKM : ABC :: $AS : AC$. But since (*Cons.*) $AC = 2AS$, the triangle ABC = $2AKM$; and ∴ the perpendicular MK bisects the triangle ABC.

EXERCISE 370.

Otherwise:—

(**Fig. 370, Plate XXV.**)—Let ABC be the given triangle. From B let fall the perpendicular BD on AC, and complete the rectangle $ADBE$; describe the parallelogram $AHFG$ similar to $ADBE$ and equal to the triangle ABC (VI. 25). Now, since the entire rectangle $AHFG$ = the triangle ABC, the triangle AHF = half the triangle ABC; and ∴ the perpendicular FH bisects the triangle ABC.

EXERCISE 371.

To divide a given triangle into two parts in a given ratio by a straight line parallel to a given straight line.

(Fig. 371, Plate XXV.)—Let ABC be the given triangle, ED the given straight line, and a to b the given ratio.

Divide AB in F in the given ratio of a to b, so that BF may be to FA as a to b (VI. 10); draw FG parallel to ED; find AK a mean proportional between AC and AG (VI. 13), and draw KH parallel to GF or to ED; then KH divides the triangle ABC in the manner required.

Join CF.

Because $AG : AK :: AK : AC$ (*hyp.*), $AG : AC$ in the duplicate ratio of $AG : AK$ (VI. 19, *Cor.*); but the triangle AFG : the triangle AHK in the duplicate ratio of $AG : AK$; and $\therefore$ the triangle AFG : to the triangle $AHK :: AG : AC$. Again, $AG : AC ::$ the triangle AFG : the triangle AFC (VI. 1); and $\therefore$ from equality of ratios, the triangle AFG : the triangle $AHK ::$ the triangle AFG : the triangle AFC, and hence $AHK = AFC$. From the triangle ABC take each of these triangles separately, and the remainders $BHKC$ and BFC are equal (*Ax.* 1). But $AF : FB ::$ the triangle AFC : the triangle BFC (VI. 1); and $\therefore AFC : BHKC :: AF : FB$, and consequently $AHK : BHKC :: AF : FB$.

EXERCISE 372.

To divide a given quadrilateral into two parts in a given ratio by a straight line parallel to one of its sides.

(Fig. 372, Plate XXV.)—Let $ABCD$ be the given quadrilateral, AD being the side to which the parallel is to be drawn. Produce the sides AB, DC to meet in F.

Let EG be the side of a rectangle equal to $ABCD$, and divided in H, so that $EH : HG$ as the part next to AD is to the remainder; let EG be produced to T so that GT may equal the side of a rectangle applied to the remaining side of the rectangle of which EG is one side, and equal to BFC; now, divide AFD in the ratio of $TH : HE$ by MN drawn parallel to AD (*Ex.* 371): then MN evidently divides $ABCD$ in the manner required.

EXERCISE 373.

If a straight line be drawn through any point in the line bisecting an angle and cutting the sides at equal angles, it will be the shortest line that can be drawn through the given point to cut them, and the triangle so cut off will be the least possible.

(Fig. 373, Plate XXV.)—Let the straight line AH bisect the angle BAC; and let the straight line MN be drawn through any point P in AH making equal angles, AMN and ANM, with AB and AC; then MN is the shortest straight line that can be drawn through P and cutting AB and AC, and the triangle AMN is the least possible that can be formed with any line through P.

Let DE be any other line through P meeting AB, AC in D and E. Through D, E draw DF, EG parallel to MN. Because the angle DAE is bisected by AP, $DA : AE :: DP : PE$ (VI. 3); but DA is greater than AE, $\therefore$ DP is greater than PE.

Again (VI. 4), $DM : MG :: DP : PE$; and as DP is greater than PE, DM is also greater than MG; and as MN is therefore nearer to GE than to DF, MN is less than half of GE and DF. But $GP + PE$ are (I. 20) greater than GE, and $DP + PF$ are greater than DF; and $\therefore$ MN is much less than half of DE and FG; and hence MN is less than DE. Again (*Ex.* 237, Part I.), the triangle AMN, whose base is bisected by the point P, is the least that can be formed by any line through the point P meeting.

EXERCISE 374.

If semicircles be described on the segments of the hypotenuse of a right-angled triangle, made by a perpendicular from the vertex, the segments of the sides intercepted by them will be in the triplicate ratio of the sides.

(Fig. 374, Plate XXV.)—Let ABC be a right-angled triangle; let CD be a perpendicular on AB from the vertex or right angle C; let semicircles be descriced on AD, DB, cutting the sides AC, BC in E, F; then $AE : BF :: AC^3 : BC^3$.

Join DE and DF.

Because BAC, CAD, DAE are similar triangles, $AB : AC :: AC : AD :: AD : AE$; and $\therefore$ $AB : AE :: AB^3 : AC^3$ (V. *Def.* 11.). Again, because the triangles ABC, DBC, DBF are similar, $AB : BC :: BC : BD :: BD : BF$; and $\therefore$ $AB : BF :: AB^3 : BC^3$.

Now, $AB : AE :: AB^3 : AC^3$; and $BF : AB :: BC^3 : AB^3$ (by inversion); and, by multiplication, $AB \cdot BF : AE \cdot AB :: AB^3 \cdot BC^3 : AC^3 \cdot AB^3$; and $\therefore$ $BF : AE :: BC^3 : AC^3$.

EXERCISE 375.

On AB, AC, two adjacent sides of a rectangle, two similar triangles are constructed, and perpendiculars are drawn to AB, AC from the angles which they subtend, intersecting at the point P. If AB, AC be homologous sides, show that P is, in all cases, in one of the diagonals of the rectangle.

(Fig. 375, Plate XXV.)—Let *AEB*, *AFC* be similar triangles described on *AB*, *AC*, two adjacent sides of the rectangle *ABDC*; let *EG* and *FH* be perpendiculars from *E* and *F* on *AB* and *AC*, meeting when produced in *P*; then *P* is, in all cases, in one of the diagonals of the rectangle.

Because *ABE* and *ACF* are similar triangles (*hyp.*), $BA : AE :: CA : AF$, and, alternately, $BA : CA :: AE : AF$, the angles around *A* being equal; but, in the right-angled similar triangles *EAG* and *FAH*, $AE : AG :: AF : AH$, and, alternately, $AE : AF :: AG : AH$; and, $\therefore$, by equality of ratios, $BA : AC :: AG : AH$, and hence the rectangles *AGPH* and *ABDC* are (**VI. 26**) about the same diagonal *AD*.

EXERCISE 376.

To find, by a geometrical construction on arithmetic, geometric and harmonic means between two given lines AB, BC.

(Fig. 376, Plate XXVI.)—Let *AB*, *BC* be in a straight line, and on *AC* describe a semicircle, and let *BD*, drawn at right angles to *AC*, meet the arc at *D*; join *D* with the centre *S*, and draw *BM* perpendicular to *DS*: then *DS*, *DB*, and *DM* are respectively the means required.

Because $DS =$ half *AB*, *BC*, it is $\therefore$ the arithmetic mean, and since $AB \cdot BC = DB^2$, *DB* is the geometric mean. Again, since $DS : DB :: DB : DM$, *DM* is the harmonic mean, being a third proportional to the arithmetic and geometric means.

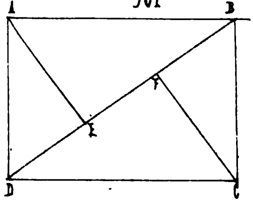
Note.—This proves geometrically the arithmetic mean is greater than the geometric, *DS* being greater than *DB* (**I. 19**).

EXERCISE 377.

In any triangle the intersections of perpendiculars from the angles on the sides, of lines from the angles bisecting the sides, and of perpendiculars bisecting the sides, are in one line; and their distances from one another are as 1, 2, 3.

PLATE XXV.

361



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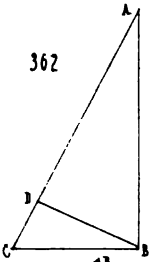

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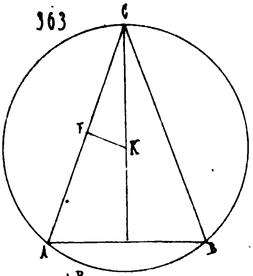

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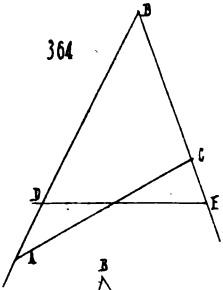

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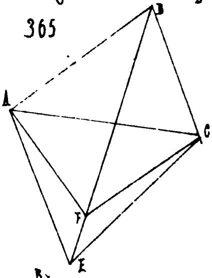

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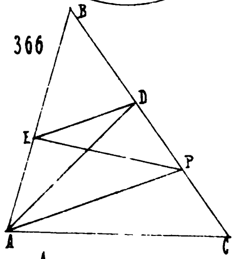

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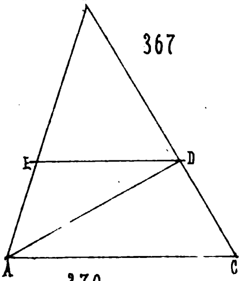

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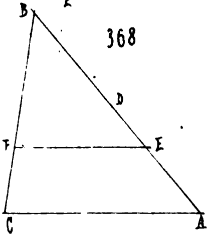

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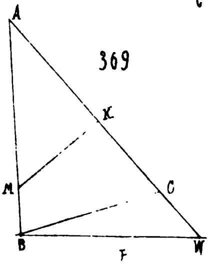

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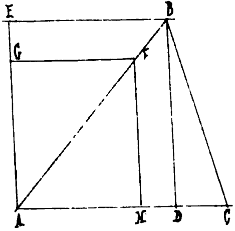

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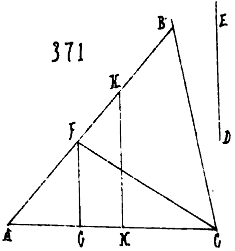

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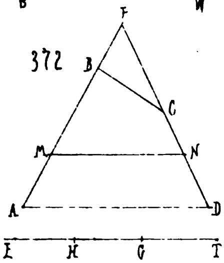

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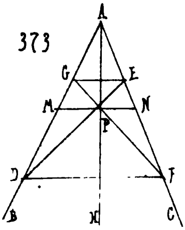

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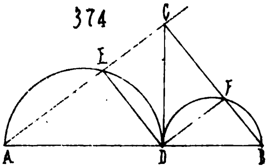

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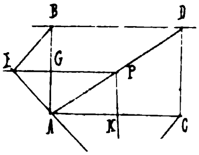
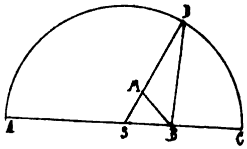


PLATE XXVI.

376



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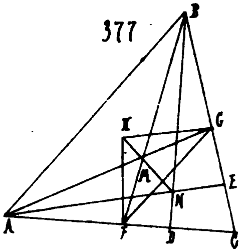

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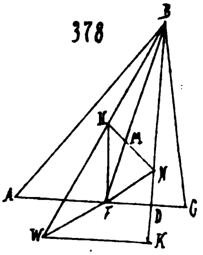

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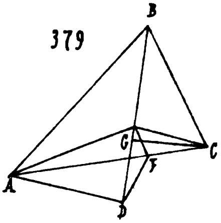

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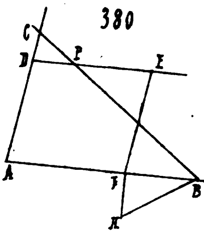

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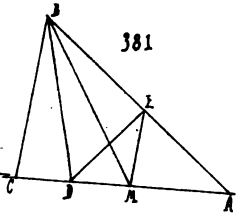

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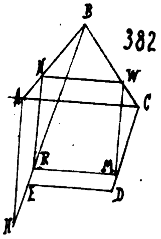

(Fig. 377, Plate XXVI.)—Let ABC be any plane triangle, and let BD, AE be two perpendiculars from B and A on the opposite sides; let BF, AG be lines from B, A bisecting the opposite sides; and FH, GH be perpendiculars bisecting the sides AC, BC; then the points N, M, H are in one straight line, and their distances are as 1, 2, 3.

Join F and G. Then FG is parallel to AB and equal to its half (*Ex.* 124, Part I.); and since AE and GH are both perpendicular to BC, they are parallel; and since BD and FH are both perpendicular to AC, they are also parallel: therefore the sides of the triangles ABN and GFH are parallel, and these triangles similar, and $AB : BN :: GF : FH$, and, alternately, $AB : GF :: BN : FH$. But $AB = 2GF$, and $\therefore BN = 2FH$. Again, the angle MBN = angle MFH (I. 29), and $MB = 2MF$ (*Ex.* 163, Part I.), $\therefore MB : BN :: MF : FH$, and the triangles MBN and MFH are similar (VI. 6), and the angle BMN = angle FMH, and as BMF is a straight line, so also is HMN. Now, as $BN = 2FH$, $MN = 2MH$, and $HN = 3MH$.

EXERCISE 378.

ANOTHER METHOD.

(Fig. 378, Plate XXVI.)—Let BD be one of the perpendiculars, and N its point of intersection with the others.

Produce BD to K making $DK = DN$; draw BW, a diameter of the circumscribing circle, and join WK, WN, the latter cutting AC in F; bisect BW in H and join FH, FB, HN, the two latter intersecting in M.

Now, since H is the point of concourse of the perpendiculars from the middle points of the sides, being the centre of circumscribing circle, it only remains to be proved that F, M are the points of bisection and trisection of AC, BF respectively.

Because $DK = DN$, and BW is the diameter of circumscribing circle, K is in the circumference of aforesaid circle, and WKB is a right angle (*Ex.* 231 and III. 31); hence WK, AC are parallel, and since AC is drawn through the middle point of KN, WN is bisected by it in F (*Ex.* 133, Part I.), and because F, H are the middle points of NW, BW, FH is parallel to BD, and $\therefore$ perpendicular to AC; and since H is the centre of circumscribing circle, AC is bisected at F (III. 3), and in triangle WBN the medians BF, NH intersect at the point of trisection.

Cor.—BN is double of FH. *Vide* (*Ex.* 124, Part I.).

EXERCISE 379.

If perpendiculars be let fall on the bisectors of vertical angle from the extremities of the base, prove that the rectangle then will be equal to the rectangle under the segments of the base made by inscribed circle.

(Fig. 379, Plate XXVI.)—Let ABC be a triangle, AD, CG perpendiculars on the bisector of the vertical angle ABC, and E, in the bisector BD, the centre of inscribed circle; let F be the point in which this circle touches AC: then $AD \cdot CG = AF \cdot CF$.

Join AE, CE, FE, FG, FD.

Because the angles ADE, AFE, CFE, CGE are right, $ADFE$, $CFGE$ are circumscribable by circles, $\therefore$ $AFD = AED$, and $CGF = CEF$ (**III. 21**), but $AED = CEF$ (*Ex.* 258), $\therefore AFD = CGF$, and $FAD = FCG$ (**I. 29**), hence FAD, FCG are similar triangles (**VI. 4**), and $AD : AF :: CF : CG$, $\therefore AD \cdot CG = AF \cdot CF$ (**VI. 16**).

EXERCISE 380.

Through a given point, between two straight lines containing a given angle, to draw a line which shall cut off a triangle equal in area to a given figure.

(Fig. 380, Plate XXVI.)—Let AB and AC be the lines containing the given angle, and P the given point.

Through P draw DE parallel to AB, and construct the parallelogrom $ADEF$ equal to the given figure (**I. 45**). Draw FH perpendicular to AB and equal to DP; make HB equal to PE; join BP, and produce it to meet AC in C; then the line BC will cut off the triangle ABC equal to the given figure.

Because the triangles CDP, EGP, BFG are similar, they are to one another in the duplicate ratio of the sides DP, PE, BF (**VI. 19**); but $HB^2 = HF^2 + FB^2$; and $\therefore$ $PE^2 = DP^2 + FB^2$, and hence the triangle EGP = the triangle CDP + triangle BFG; and $\therefore$ the triangle ABC = the parallelogram $ADEF$ = the given rectilineal figure.

EXERCISE 381.

The areas of triangles which have one angle in each equal, are as the products of the sides containing these angles.

(Fig. 381, Plate XXVI.)—Let the triangles ABC, ADE be placed so that the sides about the equal angles may coincide, E, D being in AB, AC respectively; join BD, and draw EM parallel to it meeting AC in M; join BM: then $ABC : AED :: AC \cdot AB : AD \cdot AE$.

Now, triangles AED, ABM are equal, and since they have an equal angle, $AD \cdot AE = AB \cdot AM$ (VI. 15). Again, $ABC : ABM :: AC : AM$ (VI. 1), $\therefore ABC : AED :: AC : AM :: AC \cdot AB : AM \cdot AB$; but $AD \cdot AE = AB \cdot AM$, $\therefore ABC : AED :: AC \cdot AB : AD \cdot AE$.

OTHERWISE :—

$\dfrac{ABC}{AED} = \dfrac{AC \cdot AB \, \frac{1}{2} \text{ sine } A}{AD \cdot AE \, \frac{1}{2} \text{ sine } A}$, $\therefore ABC : AED :: AC \cdot AB : AD \cdot AE$. Hence :—If the triangles be similar, then the rectangle under any two sides in one is to the rectangle under the homologous sides in the other, as the square of the third side of the former to the square of the third side of the latter.

EXERCISE 382.

To inscribe a square in a regular pentagon.

(Fig. 382, Plate XXVI.)—Let $ABCDE$ be a regular pentagon. Join AC and from A draw AH perpendicular and equal to AC; join BH, and from R, where it cuts AE, draw RN parallel to AH; draw NW and RM parallel to AC; join MW; then $NWMR$ is the square required.

Because NR is parallel to AH, $RN : AH :: NB : AB$ (VI. 2); and since NW is parallel to AC, $NW : AC :: NB : AB$; and $\therefore$ (V. 11) $RN : AH :: NW : AC$; and, alternately, $RN : NW :: AH : AC$; and since $AH = AC$, $RN = NW$. Again, since $AB = CB$, and NW is parallel to AC, $AN = WC$ (VI. 2); and since $CD = AE$, and RM is parallel to AC and ED, $ER = DM$, and $\therefore$ (*Ax.* 3), $AR = CM$, and hence, in the triangles NAR, WCM, NA and AR and the angle NAR are respectively equal to WC and CM and the angle WCM, $\therefore NR = WM$ (I. 4), and the angle ANR = angle CWM, and hence NR is parallel to WM; hence the parallelogram $RNWM$ is equilateral and rectangular, the angle RNW being a right angle, and is therefore a square.

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2